British Journal of Mathematics & Computer Science
15(2): 1-9, 2016, Article no.BJMCS.22885
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Some Inequalities in Quasi 2-normed Space $L^p(\mu), 0 < p < 1$

Risto Malčeski[1*], Vesna Manova-Erakovic[2] and Aleksa Malčeski[3]

[1]*Faculty of Informatics, FON University, Bul. Vojvodina bb, Skopje, Macedonia.*
[2]*Faculty of Natural Sciences and Mathematics, Ss. Cyril and Methodius University, Skopje, Macedonia.*
[3]*Faculty of Mechanical Engineering, Ss. Cyril and Methodius University, Skopje, Macedonia.*

Authors' contributions

This work was carried out in collaboration between all authors. Author RM designed the study, wrote the protocol and supervised the work. Authors RM and AM wrote the first draft of the manuscript. Author VME managed the literature searches and edited the manuscript. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/22885
Editor(s):
(1) Metin Basarir, Department of Mathematics, Sakarya University, Turkey.
Reviewers:
(1) Ette Harrison Etuk, Rivers State University of Science and Technology, Nigeria.
(2) Ali Mutlu, Celal Bayar University, Turkey.
(3) Mehmet Kir, Sirnak University, Turkey.
(4) Anonymous, Mexico.
Complete Peer review History: http://sciencedomain.org/review-history/13589

Received: 2nd November 2015
Accepted: 22nd February 2016
Published: 8th March 2016

Original Research Article

Abstract

In [1], C. Park generalized the notion of the quasi-normed space, i.e. he introduced the notion of the quasi 2-normed space. He proved several properties of the quasi 2-norm. In [2], M. Kir and M. Acikgoz gave the procedure of completing the quasi 2-normed space. Several inequalities relating the quasi 2-normed spaces are given in [3,4,5]. Later, in [6], some properties of the convergent sequences in quasi 2-normed spaces are proven. In this paper, we will introduce the quasi 2-norm of the space $L^p(\mu)$, $0 < p < 1$ and we will prove several inequalities relating the quasi 2-norm of this space.

Keywords: Quasi 2-norm; $(2, p)-$*norm; modul of concavity.*

2010 mathematics subject classification: 46B20.

**Corresponding author: E-mail: risto.malcheski@gmail.com;*

1 Introduction

In 1965, S. Gäh ler introduced the 2-normed spaces [7], and in [8], when $n=2$, is introduced 2-norm in the space $L^1(\mu)$. One of the axioms of the 2-norm is the inequality of parallelepiped which is fundamental in the theory of 2-normed space [4]. As in the normed spaces, C. Park, in the 2-normed spaces, replaces this inequality with a new condition, which actually gives the following definition of the quasi 2-normed space.

Definition 1 ([1]). Let L be a real vector space with $\dim L \geq 2$. *Quasi* 2-*norm* is a real function $\|\cdot,\cdot\|: L\times L \to [0,\infty)$ which satisfies:

a) $\|x,y\|\geq 0$, for all $x,y\in L$ and $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent;
b) $\|x,y\|=\|y,x\|$, for all $x,y\in L$;
c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in \mathbf{R}$, and
d) There exists a constant $K\geq 1$ such that $\|x+y,z\|\leq K(\|x,z\|+\|y,z\|)$, for all $x,y,z\in L$.

The pair $(L,\|\cdot,\cdot\|)$ is called *quasi* 2-*normed space*. The smallest number K that satisfies the condition *d*) is called the *modul of concavity* of the quasi 2-norm $\|\cdot,\cdot\|$.

In [2], M. Kir and M. Acikgoz give several examples of trivial quasi 2-normed spaces and they consider the question of completing the quasi 2-normed space, and in [1], C. Park gives a characterization of the quasi 2-normed space, i.e. he proved the following theorem.

Theorem 1 ([1]). Let $(L,\|\cdot,\cdot\|)$ be a quasi 2-normed space. Then there exists $p,\ 0<p\leq 1$ and an equivalent quasi 2-norm $|||\cdot,\cdot|||$ on L such that

$$|||x+y,z|||^p \leq |||x,z|||^p + |||y,z|||^p, \tag{1}$$

for all $x,y,z\in L$. ■

In connection with this theorem 1, C. Park, in [1], gives the following definition.

Definition 2 ([1]). The quasi 2-norm of theorem 1 is called $(2,p)-$*norm* and the quasi 2- normed space L is called $(2,p)-$*normed space*.

2 The Quasi 2-norm of $L^p(\mu),\ 0<p<1$

In this part, we will introduce the quasi 2-norm of the space $L^p(\mu), 0<p<1$. For that purpose, we will use the following two well known inequalities.

Lemma 1. If $p\in(0,1)$ and $a,b\geq 0$, then

$$(a+b)^p \leq a^p + b^p, \tag{2}$$

and equality holds if and only if $a=0$ or $b=0$. ■

Lemma 2. If $q>1$ and $c,d>0$, then

$$2^{1-q}(c+d)^q \le c^q + d^q, \tag{3}$$

and equality holds if and only if $c=d$.

Theorem 1. Let (Y,M) be a measurable space, μ be a positive measure on M and let $X=L^p(\mu)$, $p\in(0,1)$ be the space

$$X=\{f \mid f:Y\to\mathbf{R}, \int_Y |f|^p \, d\mu<+\infty\}.$$

Then the function $\|\cdot,\cdot\|: L^p(\mu)\times L^p(\mu)\to\mathbf{R}$, defined by

$$\|f,g\|=\{\int_{Y\times Y} \left|\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y)\end{vmatrix}\right|^p d(\mu\times\mu)\}^{\frac{1}{p}}, \tag{4}$$

where $\mu\times\mu$ is the direct product of the measure μ, is the quasi 2-norm of the space $X=L^p(\mu)$.

Proof. a) Because of the inequality (2), we have

$$\begin{aligned}\left|\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y)\end{vmatrix}\right|^p &= |f(x)g(y)-g(x)f(y)|^p \\ &\le (|f(x)g(y)|+|g(x)f(y)|)^p \\ &\le |f(x)g(y)|^p+|g(x)f(y)|^p,\end{aligned}$$

for all $f,g\in L^p(\mu)$ and for each $(x,y)\in Y\times Y$. Now, using (4), the properties of the absolute value and the properties of integrating on a direct product of the measurable spaces $(Y\times Y, M\times M, \mu\times\mu)$, we obtain

$$\begin{aligned}\|f,g\| &= \{\int_{Y\times Y} \left|\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y)\end{vmatrix}\right|^p d(\mu\times\mu)\}^{\frac{1}{p}} \\ &\le \{\int_{Y\times Y} (|f(x)g(y)|^p+|g(x)f(y)|^p) d(\mu\times\mu)\}^{\frac{1}{p}} \\ &= \{\int_{Y\times Y} |f(x)|^p|g(y)|^p \, d(\mu\times\mu) + \int_{Y\times Y} |g(x)|^p|f(y)|^p \, d(\mu\times\mu)\}^{\frac{1}{p}} \\ &= \{\int_Y |f(x)|^p \, d\mu \int_Y |g(y)|^p \, d\mu + \int_Y |g(x)|^p \, d\mu \int_Y |f(y)|^p \, d\mu\}^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}}\{\int_Y |f(x)|^p \, d\mu\}^{\frac{1}{p}}\{\int_Y |g(y)|^p \, d\mu\}^{\frac{1}{p}} < +\infty,\end{aligned}$$

which means that the function $\|\cdot,\cdot\|: L^p(\mu)\times L^p(\mu)\to\mathbf{R}$ is well defined.

b) Let $f,g\in L^p(\mu)$. From the definition (4) directly follows that $\|f,g\|\geq 0$. Let $\|f,g\|=0$, for some $f,g\in L^p(\mu)$, $f\neq 0, g\neq 0$ in respect of the measure μ, i.e.

$$\{\int_{Y\times Y}|\begin{vmatrix} f(x) & f(y)\\ g(x) & g(y)\end{vmatrix}|^p\, d(\mu\times\mu)\}^{\frac{1}{p}}=0.$$

Then

$$\begin{vmatrix} f(x) & f(y)\\ g(x) & g(y)\end{vmatrix}=0$$

almost everywhere in respect to the measure $\mu\times\mu$, which means that $f(x)g(y)-g(x)f(y)=0$ almost everywhere in respect of the measure $\mu\times\mu$. Therefore,

$$W_{f,g}=\{(x,y)\,|\,f(x)g(y)-g(x)f(y)\neq 0\}$$

is a set with measure 0, i.e. $(\mu\times\mu)(W_{f,g})=0$. The assumption $f\neq 0$, $g\neq 0$ in respect of the measure μ, implies that

$$U_f=\{x\,|\,x\in Y,\ f(x)\neq 0\}\text{ and }U_g=\{x\,|\,x\in Y,\ g(x)\neq 0\}$$

are sets with positive measure i.e. $\mu(U_f)>0$ и $\mu(U_g)>0$.

We will prove that

$$\mu(U_f\cap U_g)>0,\ \mu(U_f\setminus U_g)=0\text{ and }\mu(U_g\setminus U_f)=0.$$

Let us suppose that $\mu(U_f\cap U_g)=0$. Then, without loss of generality, we can assume that $U_f\cap U_g=\varnothing$ in respect of the measure μ. For every $(x,y)\in U_f\times U_g$ we have that $f(x)\neq 0, g(y)\neq 0, g(x)=0$ and $f(y)=0$, so

$$f(x)g(y)-g(x)f(y)=f(x)g(y)\neq 0.$$

Therefore $(x,y)\in W_{f,g}$, i.e. $U_f\times U_g\subseteq W_{f,g}$. Using the properties of the measure $\mu\times\mu$, i.e. from its definition we get that

$$\begin{aligned}(\mu\times\mu)(W_{f,g})&\geq(\mu\times\mu)(U_f\times U_g)\\ &=\mu(U_f)\mu(U_g)>0,\end{aligned}$$

which contradicts $(\mu\times\mu)(W_{f,g})=0$. So, we conclude that $\mu(U_f\cap U_g)>0$.

Let us now suppose that $\mu(U_f\setminus U_g)>0$. If $(x,y)\in(U_f\setminus U_g)\times(U_f\cap U_g)$, then $x\in U_f$, $x\notin U_g$, $y\in U_f$ and $y\in U_g$, so $f(x)\neq 0, g(x)=0, f(y)\neq 0$ and $g(y)\neq 0$, which implies that

$$f(x)g(y)-g(x)f(y)=f(x)g(y)\neq 0.$$

Therefore $(U_f \setminus U_g)\times(U_f \cap U_g) \subseteq W_{f,g}$. Using this and the fact that $\mu(U_f \cap U_g) > 0$, we get that

$$\begin{aligned}(\mu\times\mu)(W_{f,g}) &\geq (\mu\times\mu)[(U_f \setminus U_g)\times(U_f \cap U_g)] \\ &= \mu(U_f \setminus U_g)\mu(U_f \cap U_g) > 0,\end{aligned}$$

which, again, contradicts $(\mu\times\mu)(W_{f,g}) = 0$. We conclude that $\mu(U_f \setminus U_g) = 0$. Analogously, it can be proved that $\mu(U_g \setminus U_f) = 0$.

The equalities

$$U_f = (U_f \setminus U_g)\cup(U_f \cap U_g),\ U_g = (U_g \setminus U_f)\cup(U_f \cap U_g),$$
$$\mu(U_f \setminus U_g) = 0 \text{ and } \mu(U_g \setminus U_f) = 0,$$

imply that $U_f = U_g = U$, in respect of the measure μ.

Clearly, $\mu(U) > 0$. For each $x \in U$, we define the set

$$U_x = \{y \mid y \in U,\ f(x)g(y) - f(y)g(x) \neq 0\}.$$

We will prove that there exists $x_0 \in U$ such that $\mu(U_{x_0}) = 0$. Let us suppose the contrary, i.e. that $\mu(U_x) > 0$, for each $x \in U$. $f = g = 0$ almost everywhere on $Y \setminus U$ in respect to the measure μ, implies that $f = g = 0$ almost everywhere on $Y \setminus U_x$ in respect to the measure μ, and

$$|f(x)g(y) - g(x)f(y)|^p > 0 \text{ on } U_x.$$

Then

$$\begin{aligned}h(x) &= \int_Y |f(x)g(y) - g(x)f(y)|^p \, d\mu_y \\ &= \int_{U_x} |f(x)g(y) - g(x)f(y)|^p \, d\mu_y > 0,\end{aligned}$$

which implies that

$$\int_{Y\times Y} \left|\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y)\end{vmatrix}\right|^p d(\mu\times\mu) = \int_Y h(x)d\mu = \int_U h(x)d\mu > 0,$$

which contradicts the assumption $\| f, g \| = 0$. Therefore, there must exist $x_0 \in U$ such that

$$f(x_0)g(y) - g(x_0)f(y) = 0,$$

almost everywhere on U. Since $f(x_0) \neq 0,\ g(x_0) \neq 0$ we get that $g(y) = \frac{g(x_0)}{f(x_0)} f(y)$, for each $y \in U$, which together with the fact $f \equiv 0,\ g \equiv 0$ on $Y \setminus U$, gives that $g(y) = \frac{g(x_0)}{f(x_0)} f(y)$ на Y, i.e. $g(y) = \alpha f(y)$, where $\alpha = \frac{g(x_0)}{f(x_0)}$. So, we proved the condition *i*) from the definition 1.

c) For all $f,g \in L^p(\mu)$, it follows that

$$\begin{aligned}\| f,g \| &= \{ \int_{Y\times Y} | \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} |^p \, d(\mu\times\mu) \}^{\frac{1}{p}} \\ &= \{ \int_{Y\times Y} | \begin{vmatrix} g(x) & g(y) \\ f(x) & f(y) \end{vmatrix} |^p \, d(\mu\times\mu) \}^{\frac{1}{p}} = \| g,f \|\end{aligned}$$

i.e. the condition *ii*) from the definition 1 is true.

d) Let $f,g \in L^p(\mu)$ and $\alpha \in \mathbf{R}$. Then

$$\begin{aligned}\| \alpha f,g \| &= \{ \int_{Y\times Y} | \begin{vmatrix} \alpha f(x) & \alpha f(y) \\ g(x) & g(y) \end{vmatrix} |^p \, d(\mu\times\mu) \}^{\frac{1}{p}} \\ &= |\alpha| \{ \int_{Y\times Y} | \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} |^p \, d(\mu\times\mu) \}^{\frac{1}{p}} = |\alpha| \cdot \| f,g \|\end{aligned}$$

i.e. the condition *iii*) from the definition 1 holds.

e) Using the inequality (2), and then the inequality (3), with $q = \frac{1}{p}$, we get that, for all $f,g,h \in L^p(\mu)$, it follows that

$$\begin{aligned}\| f+g,h \| &= \{ \int_{Y\times Y} | \begin{vmatrix} f(x)+g(x) & f(y)+g(y) \\ h(x) & h(y) \end{vmatrix} |^p \, d(\mu\times\mu) \}^{\frac{1}{p}} \\ &\le \{ \int_{Y\times Y} (| \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix} | + | \begin{vmatrix} g(x) & g(y) \\ h(x) & h(y) \end{vmatrix} |)^p \, d(\mu\times\mu) \}^{\frac{1}{p}} \\ &\le \{ \int_{Y\times Y} (| \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix} |^p + | \begin{vmatrix} g(x) & g(y) \\ h(x) & h(y) \end{vmatrix} |^p) d(\mu\times\mu) \}^{\frac{1}{p}} \\ &= \{ \int_{Y\times Y} | \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix} |^p \, d(\mu\times\mu) + \int_{Y\times Y} | \begin{vmatrix} g(x) & g(y) \\ h(x) & h(y) \end{vmatrix} |^p \, d(\mu\times\mu) \}^{\frac{1}{p}} \\ &\le 2^{\frac{1}{p}-1} \{ (\int_{Y\times Y} | \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix} |^p \, d(\mu\times\mu))^{\frac{1}{p}} + (\int_{Y\times Y} | \begin{vmatrix} g(x) & g(y) \\ h(x) & h(y) \end{vmatrix} |^p \, d(\mu\times\mu))^{\frac{1}{p}} \} \\ &= 2^{\frac{1}{p}-1} (\| f,h \| + \| g,h \|),\end{aligned}$$

which means that, for $K = 2^{\frac{1}{p}-1} > 1$, the condition *iv*) from the definition 1 holds.

In [1], C. Park defines Cauchy sequence, convergent sequence and complete quasi 2-normed space i.e. he gives the following definition.

Definition 3. Let L be a quasi 2-normed space.

a) The sequence $\{x_n\}_{n=1}^{\infty}$ in L is a *Cauchy* if

$\lim\limits_{n\to\infty} \| x_m - x_n, z \| = 0$, for each $z \in L$.

b) The sequence $\{x_n\}_{n=1}^{\infty}$ in L is *convergent* if there exists $x \in L$ such that

$\lim_{n\to\infty} \| x_m - x, z \| = 0$, for each $z \in L$.

The vector $x \in L$ is called the *boundary* of the sequence $\{x_n\}_{n=1}^{\infty}$.

c) A quasi 2-normed space L in which every Cauchy sequence is convergent is called a *quasi* 2-*complete* (a *quasi* 2-*Banach*) *space*.

Problem. Whether the quasi 2-normed space $L^p(\mu)$, $0 < p < 1$ with the quasi norm, introduced by (4), is complete?

3 Two Inequalities in $L^p(\mu), 0 < p < 1$ as Quasi 2-normed Space

For the quasi 2-normed spaces in [4], it is proven the following Lemma.

Lemma 3. If L is quasi 2-normed space with modul of concavity $K \geq 1$, then, for each $n > 1$ and for all $z, x_1, x_2, ..., x_n \in L$, it follows that

$$\| \sum_{i=1}^{n} x_i , z \| \leq K^{1+[\log_2(n-1)]} \sum_{i=1}^{n} \| x_i , z \| . \blacksquare \tag{5}$$

From Lemma 3 and the proof of Theorem 1, directly follows the consequence.

Corollary 1. Let $L^p(\mu), 0 < p < 1$ be a quasi 2-normed space in which the quasi 2-norm is given by (4). Then for all $f_1, f_2, ..., f_n, g \in L^p(\mu)$, $n \geq 2$, it follows that

$$\| \sum_{i=1}^{n} f_i , g \| \leq 2^{(1+[\log_2(n-1)])(\frac{1}{p}-1)} \sum_{i=1}^{n} \| f_i , g \| . \tag{6}$$

Proof. It is enough to take $K = 2^{\frac{1}{p}-1} > 1$ in the inequality (5). $\blacksquare$

In the following theorem, we will prove stricter inequality then the inequality (6). For that purpose, we will use the following well known inequality.

Lemma 4. Let $t \in [1, \infty)$. Then for all $a_1, a_2, ..., a_n \in [0, \infty)$, it follows that

$$(\sum_{i=1}^{n} a_i)^t \leq n^{t-1} \sum_{i=1}^{n} a_i^t . \blacksquare$$

Theorem 2. Let $L^p(\mu), 0 < p < 1$ be a quasi 2-normed space in which the quasi 2-norm is given by (4). Then for all $f_1, f_2, ..., f_n, g \in L^p(\mu)$, it follows that

$$\| \sum_{i=1}^{n} f_i , g \| \leq n^{\frac{1}{p}-1} \sum_{i=1}^{n} \| f_i , g \| . \tag{7}$$

Proof. Since $t=\frac{1}{p}>1$, the inequality (4), the properties of the integral, Lemma 1 and Lemma 4 imply that

$$\begin{aligned}
\|\sum_{i=1}^{n} f_i, g\| &= \{\int_{Y\times Y} |\sum_{i=1}^{n}\begin{vmatrix} f_i(x) & f_i(y)\\ g(x) & g(y)\end{vmatrix}|^p \, d(\mu\times\mu)\}^{\frac{1}{p}}\\
&\leq \{\int_{Y\times Y} (\sum_{i=1}^{n}|\begin{vmatrix} f_i(x) & f_i(y)\\ g(x) & g(y)\end{vmatrix}|)^p \, d(\mu\times\mu)\}^{\frac{1}{p}}\\
&\leq \{\int_{Y\times Y} \sum_{i=1}^{n}|\begin{vmatrix} f_i(x) & f_i(y)\\ g(x) & g(y)\end{vmatrix}|^p \, d(\mu\times\mu)\}^{\frac{1}{p}}\\
&= \{\sum_{i=1}^{n}\int_{Y\times Y} |\begin{vmatrix} f_i(x) & f_i(y)\\ g(x) & g(y)\end{vmatrix}|^p \, d(\mu\times\mu)\}^{\frac{1}{p}}\\
&\leq n^{\frac{1}{p}-1}\sum_{i=1}^{n}\{\int_{Y\times Y} |\begin{vmatrix} f_i(x) & f_i(y)\\ g(x) & g(y)\end{vmatrix}|^p \, d(\mu\times\mu)\}^{\frac{1}{p}}\\
&= n^{\frac{1}{p}-1}\sum_{i=1}^{n}\| f_i, g\|.
\end{aligned}$$

Clearly, the inequality (7) is stricter than the inequality (6), since for each $n\geq 2$, it holds $n\leq 2^{1+[\log_2(n-1)]}$, and the equality holds for $n=2^k, k=1,2,3,\dots$.

To prove that $n^{\frac{1}{p}-1}$ is the best possible constant, it is enough to take that E and F are measurable sets such that $\mu(E)=\alpha<\infty, \mu(F)=\beta<\infty$, $g=1_F$ and $E_i=E$ and $f_i=1_{E_i}$, for $i=1,2,\dots,n$. Then

$$\begin{aligned}
\|\sum_{i=1}^{n} f_i, g\| &= \{\sum_{i=1}^{n}(\mu\times\mu)(E_i\times F)\}^{\frac{1}{p}} = \{\sum_{i=1}^{n}\mu(E_i)\mu(F)\}^{\frac{1}{p}} = (n\alpha\beta)^{\frac{1}{p}}\\
&= n^{\frac{1}{p}-1}\{n(\alpha\beta)^{\frac{1}{p}}\} = n^{\frac{1}{p}-1}\sum_{i=1}^{n}\{\mu(E_i)\mu(F)\}^{\frac{1}{p}}\\
&= n^{\frac{1}{p}-1}\sum_{i=1}^{n}(\mu\times\mu)(E_i\times F)^{\frac{1}{p}} = n^{\frac{1}{p}-1}\sum_{i=1}^{n}\| f_i, g\|. \ \blacksquare
\end{aligned}$$

4 Conclusion

In this paper, it's introduced the quasi 2-norm of the space $L^p(\mu)$, $0<p<1$ and there are proven several inequalities relating the quasi 2-norm of this space. Also is given the following open problem:

> Whether the quasi 2-normed space $L^p(\mu)$, $0<p<1$ with the quasi norm, introduced by (4), is complete?

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Park C. Generalized quasi-banach spaces and quasi-(2;p) normed spaces. Journal of the Chungcheong Matematical Society. 2006;19(2):197–206.

[2] Kir M, Acikgoz M. A study involving the completion of quasi 2-normed space. International Journal of Analysis; 2013.
Available: http://dx.doi/10.1155.2013/ 512372

[3] Anevska K, Malčeski S. Inequalities of Dunkl-Williams and Mercer type in quasi 2-normed space. Matematichki Bilten. 2015;1:33–37.

[4] Malčeski A, Malčeski R, Anevska K, Malčeski S. A remark about quasi 2-normed space. Applied Mathematical Sciences. 2015;9(55):2717–2727.

[5] Malčeski A, Malčeski R, Inequalities of Pečarić-Rajić type in quasi 2-normed space. IJSIMR. 2015;3(5):1–7.

[6] Malčeski R. Several properties of convergent and Cauchy sequences in a quasi 2-normed space. Matematichki Bilten. 2015;1:19–24.

[7] Gähler S. Lineare 2-normierte Räume. Math. Nachr. 1965;28:1–42.

[8] Malčeski A, Malčeski R. $L^1(\mu)$ as a *n*-normed spaces. Annuaire de l'Institut des Mathématiques. 1997;Tome 38:23-30.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/13589