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Solving the Space-time Fractional Burger's and the Fifth-order Sawda-Kotera Equations Using the Fractional Sub-equation Method

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, a fractional sub-equation method is proposed for finding exact solutions of the space–time fractional Burger's equation and the space-time fractional fifth-order Sawda-Kotera equation. The derivative is defined in the Jumarie's modified Riemann-Liouville sense. The proposed method is based on fractional Riccati's equation. Accordingly, it was obtained three different exact solutions, namely the generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solutions. The proposed scheme can also be applied to other nonlinear fractional partial differential equations.

Keywords: Fractional differential equation; fractional sub-equation method; modified Riemann-Liouville derivative; Burger's equation; Sawda-Kotera equation; Mittag-Leffler function; analytical solutions.

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1 Introduction

It is known that fractional differential equations are generalizations of classical differential equations of integer order. Nonlinear fractions partial differential equations (NFPDEs) are used to describe many phenomena in several areas such physics, engineering, biology [1-11].

Existence, uniqueness and stability of solutions of NFPDEs have been investigated by many authors (see for example [12-16]).

In the last decade, many analytical and numerical methods have been proposed to obtain solutions of NFPDEs, such as fractional functional variable method [17], Legendre spectral-collocation method [18-20], finite difference method [21-23], finite element method [24-26], differential transform method [27-29], homotopy analysis method [30,31] and so on [32-36].

Finally, the Simplified bilinear method and the (G'/G) expansion method are used to obtain exact solutions of some important fractional nonlinear equations arise in physics [37-39].

In 2011, Zhang et al. [40] proposed an important new method called fractional sub-equation method to find for traveling wave solutions of NFPDEs. This new method depends on the homogeneous balance principle and Jumarie's modified Riemann-Liouville derivative [41]. By using the fractional sub-equation method, Alzaidy J. [42] solved two nonlinear space-time FPDEs. Zhang et al.'s work was improved by Guo et al. [43] and Lu [44] to obtain exact solutions of some nonlinear space-time FPDEs.

In this paper, we will apply the fractional sub-equation method for solving fractional partial differential equations in the sense of modified Riemann-Liouville derivative by Jumarie [41]. To demonstrate the validity and advantages of the method, we will apply it to the space–time fractional Burger's equation and the space-time fractional fifth-order Sawda-Kotera equation.

This proposed method to our knowledge has not been applied to the space–time fractional Burger's equation and the space-time fractional fifth-order Sawda-Kotera equation.

The article is organized as follows: In Section 2, we will describe the Jumarie's modified Riemann-Liouville derivative with some of its important properties and give the main steps of the method here. In Section 3, we will give two applications of the proposed method to NFPDEs. Finally in Section 4, some conclusions are given.

2 Description of Modified Riemann-Liouville Derivative and the Proposed Method

The Jumarie's modified Riemann-Liouville derivative of order α is defined by the expression [41]

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} (f(\xi) - f(0)) d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ \left[f^{(\alpha-n)}(x) \right]^{(n)} & n \le \alpha < n+1,\ n \ge 1. \end{cases} \tag{1}$$

Here we summarize some properties for the proposed modified Riemann–Liouville derivative are listed in [41] as follows:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0, \tag{2}$$

$$D_x^\alpha (f(x)g(x)) = g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x), \tag{3}$$

$$D_x^\alpha f[g(x)] = f_g'[g(x)] D_x^\alpha g(x) = D_g^\alpha f[g(x)](g'(x))^\alpha, \tag{4}$$

which will be used in the following sections.

We show the main steps of the fractional sub-equation method as follows.

Step 1. Assume that we have the following NFPDE in the form:

$$F(u, u_x, u_t, D_x^\alpha u, D_t^\alpha u, \ldots) = 0, \quad 0 < \alpha \le 1, \tag{5}$$

where $D_x^\alpha u$ and $D_t^\alpha u$ are defined in (1), F is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 2. Using the following wave transformation:

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \tag{6}$$

where k and c are constants to be determined later, the NFPDE (5) is reduced to the following nonlinear fractional ordinary differential equation (NFODE) for $u = u(\xi)$:

$$F(u, ku', cu', k^\alpha D_\xi^\alpha u, c^\alpha D_\xi^\alpha u, \ldots) = 0. \tag{7}$$

Step 3. Suppose that Eq. (7) has the formal solution:

$$u(\xi) = \sum_{i=0}^{n} a_i \varphi^i, \tag{8}$$

where a_i $(i = 0, 1, 2, \ldots, n)$ are constants to be determined later, n is a positive integer determined by balancing the highest order derivatives and nonlinear terms in Eq. (5)or Eq. (7) (see [45] for details), whereas $\varphi = \varphi(\xi)$ satisfies the following fractional Riccati's equation:

$$D_\xi^\alpha \varphi(\xi) = \sigma + \varphi^2(\xi), \tag{9}$$

where σ is a real constant. By using the generalized Exp-function method via Mittag-Leffler functions [46], Zhang et al. presented the following solutions of fractional Riccati's equation (9)

$$\varphi(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh_{\alpha}(\sqrt{-\sigma}\,\xi), & \sigma<0,\\ -\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\,\xi), & \sigma<0,\\ \sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\,\xi), & \sigma>0,\\ -\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\,\xi), & \sigma>0,\\ -\dfrac{\Gamma(1+\alpha)}{\xi^{\alpha}+\omega}, \quad \omega=\text{const.}, & \sigma=0,\end{cases} \tag{10}$$

where the generalized hyperbolic and trigonometric functions are defined as

$$tanh_{\alpha}(\xi)=\frac{E_{\alpha}(\xi^{\alpha})-E_{\alpha}(-\xi^{\alpha})}{E_{\alpha}(\xi^{\alpha})+E_{\alpha}(-\xi^{\alpha})}, \qquad coth_{\alpha}(\xi)=\frac{E_{\alpha}(\xi^{\alpha})+E_{\alpha}(-\xi^{\alpha})}{E_{\alpha}(\xi^{\alpha})-E_{\alpha}(-\xi^{\alpha})}, \tag{11}$$

$$tan_{\alpha}(\xi)=-i\left(\frac{E_{\alpha}(i\xi^{\alpha})-E_{\alpha}(-i\xi^{\alpha})}{E_{\alpha}(\xi^{\alpha})+E_{\alpha}(-\xi^{\alpha})}\right), \qquad cot_{\alpha}(\xi)=i\left(\frac{E_{\alpha}(i\xi^{\alpha})+E_{\alpha}(-i\xi^{\alpha})}{E_{\alpha}(\xi^{\alpha})-E_{\alpha}(-\xi^{\alpha})}\right).$$

where $E_{\alpha}(z)$ denotes the Mittag-Leffler function, given by

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(1+k\alpha)}. \tag{12}$$

where Gamma function is defined for Re(z) > 0 by

$$\Gamma(z)=\int_{0}^{\infty}\mathrm{e}^{-t}t^{z-1}\,\mathrm{d}t \tag{13}$$

Step 4. Substituting Eq. (8) in conjunction with Eq. (9) into Eq. (7) and using Eqs. (2-4), we can obtain polynomial in $\varphi(\xi)$. Status all the coefficients of $\varphi^{m}\,(m=0,1,2,\ldots)$ to zero, yields a set of nonlinear algebraic equations for $a_i\,(i=0,1,2,\ldots,n),k$ and c.

Step 5. Finally, Solving algebraic equations in step 4 for $a_i\,(i=0,1,2,\ldots,n),k$ and c by using the Maple or Mathematica, substituting these constants and the solutions of Eq.(9) into Eq.(8), we can obtain the explicit solutions of Eq.(5) immediately.

Remark: When $\alpha=1$, the fractional Riccati sub-equation becomes classical Riccati equation $\varphi'(\xi)=\sigma+\varphi^{2}(\xi)$ used in [47]. So the method in this paper can be applied to solve integer-order differential equations. We conclude that our method is a general for the tanh-function method.

3 Applications

In this section, we will apply the fractional sub-equation method to construct exact solutions for some nonlinear fractional partial differential equations (NFPDEs), namely the space–time fractional Burger's equation and the space–time fractional fifth-order Sawda-Kotera equation which are very important NFPDEs in mathematical physics and have been paid attention by many researchers.

3.1 Example 1. The space–time fractional Burger's equation

We consider the space–time fractional Burger's equation.

$$D_t^{\alpha} u + A\, D_x^{\beta} u - B\, u_{xx} + \varepsilon\, u\, u_x = 0\,.\, 0 < \alpha \le 1, 0 < \beta < 1 \tag{14}$$

where A, B and ε are constants. This equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer [48]. Now, using the traveling wave transformation (6), Eq. (14) can be reduced to the following NFODE:

$$c^{\alpha}\, D_{\xi}^{\alpha} u + A\, k^{\beta}\, D_{\xi}^{\beta} u - B\, k^2 D_{\xi}^2 u + \varepsilon\, k\, u\, D_{\xi} u = 0. \tag{15}$$

By balancing the highest order derivative terms and nonlinear terms in Eq. (15), gives the value of $n = 1$. Thus, we have

$$u(\xi) = a_0 + a_1 \varphi(\xi), \tag{16}$$

where $\varphi(\xi)$ satisfies the Riccati equations

$$\begin{cases} D_{\xi}^{\alpha} \varphi = \sigma_1 + \varphi^2, \\ D_{\xi}^{\beta} \varphi = \sigma_2 + \varphi^2. \\ D_{\xi} \varphi = \sigma_3 + \varphi^2. \end{cases} \tag{17}$$

Substituting Eq. (16) along with Eqs. (17) into Eq. (15) and then setting the coefficients of φ^i $(i = 0,1,2,3)$ to zero, we can obtain a set ofalgebraic equations for a_0, a_1, σ_1 as follows:

$$\begin{aligned} &\varphi^0 : c^{\alpha} a_1 \sigma_1 + A k^{\beta} a_1 \sigma_2 + \varepsilon k a_0 a_1 \sigma_3 = 0 \\ &\varphi^1 : \varepsilon k a_1{}^2 \sigma_3 - 2\, B k^2 a_1 \sigma_3 = 0 \\ &\varphi^2 : c^{\alpha} a_1 + A k^{\beta} a_1 + \varepsilon k a_0 a_1 = 0 \\ &\varphi^3 : -2\, B k^2 a_1 + \varepsilon k a_1{}^2 = 0 \end{aligned} \tag{18}$$

Solving the algebraic equations(18) by Mathematica, we have:

$$a_0 = -\frac{c^{\alpha} + A\, k^{\beta}}{\varepsilon\, K}, \qquad a_1 = \frac{2\, B\, k}{\varepsilon}, \qquad \sigma_1 = \frac{A\, k^{\beta}}{c^{\alpha}} (\sigma_3 - \sigma_2) + \sigma_3\,. \tag{19}$$

From Eqs. (10), (16) and (19), we get three different types of exact solutions of Eq. (14) as follows:

Two generalized hyperbolic function solutions

If $\sigma_1 = \frac{Ak^{\beta}}{c^{\alpha}} (\sigma_3 - \sigma_2) + \sigma_3 < 0$, then

$$u(\xi) = \begin{cases} -\frac{c^{\alpha} + Ak^{\beta}}{\varepsilon K} - \frac{2\, Bk}{\varepsilon} \sqrt{\frac{Ak^{\beta}}{c^{\alpha}} (-\sigma_3 + \sigma_2) - \sigma_3}\, tanh_{\alpha} \left(\sqrt{\frac{Ak^{\beta}}{c^{\alpha}} (-\sigma_3 + \sigma_2) - \sigma_3 \xi} \right), \\ -\frac{c^{\alpha} + Ak^{\beta}}{\varepsilon K} - \frac{2\, Bk}{\varepsilon} \sqrt{\frac{Ak^{\beta}}{c^{\alpha}} (-\sigma_3 + \sigma_2) - \sigma_3}\, coth_{\alpha} \left(\sqrt{\frac{Ak^{\beta}}{c^{\alpha}} (-\sigma_3 + \sigma_2) - \sigma_3 \xi} \right). \end{cases} \tag{20}$$

Two generalized trigonometric function solutions

If $\sigma_1 = \frac{A\,k^{\beta}}{c^{\alpha}}(\sigma_3 - \sigma_2) + \sigma_3 > 0$, then

$$u(\xi) = \begin{cases} -\frac{c^{\alpha}+A\,k^{\beta}}{\varepsilon\,K} + \frac{2\,B\,k}{\varepsilon}\sqrt{\frac{A\,k^{\beta}}{c^{\alpha}}(\sigma_3 - \sigma_2) + \sigma_3}\,tan_{\alpha}\left(\sqrt{\frac{A\,k^{\beta}}{c^{\alpha}}(\sigma_3 - \sigma_2) + \sigma_3}\,\xi\right), \\ -\frac{c^{\alpha}+A\,k^{\beta}}{\varepsilon\,K} - \frac{2\,B\,k}{\varepsilon}\sqrt{\frac{A\,k^{\beta}}{c^{\alpha}}(\sigma_3 - \sigma_2) + \sigma_3}\,cot_{\alpha}\left(\sqrt{\frac{A\,k^{\beta}}{c^{\alpha}}(\sigma_3 - \sigma_2) + \sigma_3}\,\xi\right). \end{cases} \tag{21}$$

One rational solution

If $\sigma_1 = \frac{Ak^{\beta}}{c^{\alpha}}(\sigma_3 - \sigma_2) + \sigma_3 = 0$, then

$$u(\xi) = -\frac{c^{\alpha}+Ak^{\beta}}{\varepsilon K} - \frac{2\,Bk\Gamma(1+\alpha)}{\varepsilon(\xi^{\alpha}+\omega)}. \tag{22}$$

As $\alpha \to 1$ and $\beta \to 1$the results obtained above become the ones of Eq. (14).

3.2 Example 2. Space-time fractional-fifth-order Sawda-Kotera equation

Next, we consider the following space-time fractional fifth-order Sawda-Kotera equation discussed in [49]:

$$D_t^{\alpha}u + D_x^{5\alpha}u + 45\,u^2 D_x^{\alpha}u + 15\,D_x^{\alpha}u D_x^{2\alpha}u + 15\,u D_x^{3\alpha}u = 0, \qquad 0 < \alpha \le 1. \tag{23}$$

Using the traveling wave transformation (6), Eq. (23) can be reduced to the following NFODE:

$$c^{\alpha}D_t^{\alpha}u + K^{5\alpha}D_x^{5\alpha}u + 45K^{\alpha}u^2 D_x^{\alpha}u + 15\,K^{3\alpha}D_x^{\alpha}u D_x^{2\alpha}u + 15\,K^{3\alpha}u D_x^{3\alpha}u = 0, \tag{24}$$

Subsequently, we assume that (24) admits a solution in the form

$$u(\xi) = \sum_{i=0}^{n} a_i \varphi^i\,, \tag{25}$$

At this stage we apply the same technique as in the case of the previous example, namely, by balancing the highest order derivative terms and nonlinear terms in (24), then substituting (25) with $n = 2$, with (9) into (24), we finally obtain the corresponding system of algebraic equations as

$$\begin{aligned}
&\boldsymbol{\varphi}^{\mathbf{0}}{:}30\,k^{3\alpha}a_1a_2\sigma+16\,k^{5\alpha}a_1\sigma^3+45\,k^{\alpha}a_0{}^2a_1\sigma+30\,k^{3\alpha}a_0a_1\sigma^2+c^{\alpha}a_1\sigma=0,\\
&\boldsymbol{\varphi}^{\mathbf{1}}{:}60\,k^{3\alpha}a_2{}^2\sigma^3+272\,k^{5\alpha}a_2\sigma^3+90\,k^{\alpha}a_0^2a_2\sigma+90\,k^{\alpha}a_0\,a_1^2\sigma+240\,k^{3\alpha}a_0\,a_2\,\sigma^2\\
&\qquad +60\,k^{3\alpha}a_1{}^2\sigma^2+2\,c^{\alpha}a_2\sigma=0,\\
&\boldsymbol{\varphi}^{\mathbf{2}}{:}136\,k^{5\alpha}a_1\sigma^2+45\,k^{\alpha}a_0^2a_1+180\,k^{\alpha}a_0\,a_1\,a_2\sigma+45\,k^{\alpha}(2a_1\,a_2+a_1{}^2)a_1\sigma\\
&\qquad +2\,c^{\alpha}a_1+480\,k^{3\alpha}a_1\,a_2\,\sigma^2+120\,k^{3\alpha}a_1\,a_2\,\sigma=0,\\
&\boldsymbol{\varphi}^{\mathbf{3}}{:}540\,k^{3\alpha}a_2^2\sigma^2+180\,k^{3\alpha}a_1^2\sigma+2c^{\alpha}a_2+600\,k^{3\alpha}a_0a_2\sigma+1232\,k^{5\alpha}a_2\,\sigma^2\\
&\qquad +90\,k^{\alpha}a_0^2a_2+90\,k^{\alpha}a_0\,a_1^2+90\,k^{\alpha}(2a_0\,a_2+a_1{}^2)a_2\sigma+90\,k^{\alpha}a_1^2a_2\,\sigma=0,\\
&\boldsymbol{\varphi}^{\mathbf{4}}{:}1050\,k^{3\alpha}a_1a_2\sigma+90\,k^{3\alpha}a_0\,a_1+240\,k^{5\alpha}a_1\,\sigma+180\,k^{\alpha}a_0\,a_1\,a_2\\
&\qquad +45\,k^{\alpha}(2a_0\,a_2+a_1{}^2)a_1+225\,k^{\alpha}a_1\,a_2{}^2\sigma=0,\\
&\boldsymbol{\varphi}^{\mathbf{5}}{:}1680\,k^{5\alpha}a_2\sigma+90\,k^{\alpha}(2a_0\,a_2+a_1{}^2)a_2+90\,k^{\alpha}a_1^2a_2+90\,k^{\alpha}a_2^3\sigma 1020\,k^{3\alpha}a_2^2\sigma\\
&\qquad +120\,k^{3\alpha}a_1^2+360\,k^{3\alpha}a_0a_2=0,\\
&\boldsymbol{\varphi}^{\mathbf{6}}{:}225\,k^{\alpha}a_1a_1^2+600\,k^{3\alpha}a_1\,a_1+120\,k^{5\alpha}a_1=0,\\
&\boldsymbol{\varphi}^{\mathbf{7}}{:}9\,k^{\alpha}a_2^3+540\,k^{3\alpha}a_2^2+720\,k^{5\alpha}a_2=0.
\end{aligned} \tag{26}$$

Solving the set of algebraic equations (26) for a_0, a_1, a_2 and σ by Maple or Mathematica yields

Table 1. The results

Case	a_0	a_1	a_2	σ
1	$-\frac{1}{15}(20k^{2\alpha}\sigma+\sqrt{5}k^{-\alpha}\sqrt{4k^{6\alpha}\sigma^2-c^\alpha k^\alpha})$	0	$-2k^{2\alpha}$	σ
2	$\frac{1}{15}(-20k^{2\alpha}\sigma+\sqrt{5}k^{-\alpha}\sqrt{4k^{6\alpha}\sigma^2-c^\alpha k^\alpha})$	0	$-2k^{2\alpha}$	σ
3	$-\frac{2}{3}ic^{\alpha/2}k^{-\alpha/2}$	0	$-4k^{2\alpha}$	$\frac{1}{4}ic^{\alpha/2}k^{-5\alpha/2}$
4	$\frac{2}{3}ic^{\alpha/2}k^{-\alpha/2}$	0	$-4k^{2\alpha}$	$-\frac{1}{4}ic^{\alpha/2}k^{-5\alpha/2}$
5	$-\frac{5}{3}ic^{\alpha/2}k^{-\alpha/2}$	0	$-2k^{2\alpha}$	$ic^{\alpha/2}k^{-5\alpha/2}$
6	$\frac{5}{3}ic^{\alpha/2}k^{-\alpha/2}$	0	$-2k^{2\alpha}$	$-ic^{\alpha/2}k^{-5\alpha/2}$
7	$-\frac{16ic^{\alpha/2}k^{-\alpha/2}}{3\sqrt{155}}$	0	$-2k^{2\alpha}$	$\frac{1}{2}i\sqrt{\frac{5}{31}}c^{\alpha/2}k^{-5\alpha/2}$
8	$\frac{16ic^{\alpha/2}k^{-\alpha/2}}{3\sqrt{155}}$	0	$-2k^{2\alpha}$	$-\frac{1}{2}i\sqrt{\frac{5}{31}}c^{\alpha/2}k^{-5\alpha/2}$
9	$-\frac{2}{3}c^{\alpha/2}k^{-\alpha/2}$	0	$-2k^{2\alpha}$	$\frac{1}{2}c^{\alpha/2}k^{-5\alpha/2}$
10	$\frac{2}{3}c^{\alpha/2}k^{-\alpha/2}$	0	$-2k^{2\alpha}$	$-\frac{1}{2}c^{\alpha/2}k^{-5\alpha/2}$

Based on the sign of σ there are at most 42 distinct non-trivial solutions provided $k \neq 0 \neq c$ Finally, from Eqs. (10), (25) with $n = 2$ and Table 1 we obtain the following generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solution of Eq. (23). For example in case 1:

$$u(\xi)=\begin{cases} \frac{1}{15}\left(20k^{2\alpha}\sigma+\sqrt{5}k^{-\alpha}\sqrt{4k^{6\alpha}\sigma^2-c^\alpha k^\alpha}\right)+2k^{2\alpha}\sigma\, tanh_\alpha^2\left(\sqrt{-\sigma}\xi\right), & \sigma<0, \\ -\frac{1}{15}\left(20k^{2\alpha}\sigma+\sqrt{5}k^{-\alpha}\sqrt{4k^{6\alpha}\sigma^2-c^\alpha k^\alpha}\right)+2k^{2\alpha}\sigma\, coth_\alpha^2\left(\sqrt{-\sigma}\xi\right), & \sigma<0, \\ -\frac{1}{15}\left(20k^{2\alpha}\sigma+\sqrt{5}k^{-\alpha}\sqrt{4k^{6\alpha}\sigma^2-c^\alpha k^\alpha}\right)-2k^{2\alpha}\sigma\, tan_\alpha^2\left(\sqrt{\sigma}\xi\right), & \sigma>0, \\ -\frac{1}{15}\left(20k^{2\alpha}\sigma+\sqrt{5}k^{-\alpha}\sqrt{4k^{6\alpha}\sigma^2-c^\alpha k^\alpha}\right)-2k^{2\alpha}\sigma\, cot_\alpha^2\left(\sqrt{\sigma}\xi\right), & \sigma>0, \\ -\frac{1}{15}\left(\sqrt{5}k^{-\alpha}\sqrt{-c^\alpha k^\alpha}\right)-2k^{2\alpha}\left[\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}\right]^2, & \sigma=0, \end{cases} \tag{27}$$

where $\xi = kx + ct$.

And the remaining solutions will be in the same manner.

4 Conclusion

In this article, based on the fractional sub-equation method, we have successfully found out five exact analytical solutions for the space–time fractional Burger's equation and at most 42 exact analytical solutions for the space-time fractional fifth-order Sawda-Kotera. These solutions including the generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solutions.

From our results obtained in this paper, we conclude that the fractional sub-equation method is powerful, effective and convenient for solving NFPDEs. Also, the solutions of the proposed NFPDEs in this paper may have many potential applications in physics and engineering.

Finally, we believe that this method provides a powerful mathematical tool to obtain exact analytical solutions of a great many NFPDEs in mathematical physics.

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Competing Interests

Authors have declared that no competing interests exist.

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