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New Analytic Technique for the Solution of Nth Order Nonlinear Two-point Boundary Value Problems

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Authors' contributions

This work was carried out in collaboration between both authors. Author SOA designed the study, wrote the first and revised draft of the manuscript. Author EO implemented numerical examples and managed literature searches. Both authors read and approved the final manuscript.

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Abstract

In this article, a new analytic technique based on Parker-Sochacki iteration is introduced for computing series solution of a general nonlinear two-point boundary value problems with Dirichlet and Neumann boundary conditions. For problems with or without analytic solution, we found out that this easy-to-implement method produced highly accurate results without linearization when compared with their closed form solutions.

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1 Introduction

Non-linear boundary value problems arise naturally in the modeling of real life problems in Engineering and Biology. The computation of closed form solutions to these problems has remained elusive to researchers for many years. Recently, research efforts have been devoted to computing approximate solution to such boundary value problems. Solution approximation techniques such as homotopy perturbation method (HPM) [1], [2], homotopy analysis method (HAM)[3], variational iteration method (VIM) [4] have been developed and applied to problems in the literature. While reports abound for cases where these celebrated methods have been successfully applied, these methods nevertheless have their own shortcomings. For instance, HPM is not accessible to problems with no small parameters, successful computation of Lagrange multiplier is an unavoidable hurdle in the application of VIM. In general, it has been reported that the solution produced by all these approximation techniques for BVP in the literature are only approximations of their corresponding Maclaurin series solution [5]. For this reason, Maclaurin series solution to BVP remains important either to compute the series solution to the problem under consideration or to validate already existing results (via other approximation techniques).

In this article, we are concerned with finding analytic solution to a general two-point boundary value problem of the form

$$\frac{d^k y}{dx^k}(x) = f\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2} \dots, \frac{d^{k-1}y}{dx^{k-1}}\right), \quad a < x < b,\ k \in \mathbb{N} \tag{1.1}$$

with boundary conditions

$$\frac{d^i y}{dx^i}(a) = \alpha_i, \quad \frac{d^j y}{dx^j}(b) = \beta_i,\ \ i, j \in \{0, 1, 2, \dots, k-1\},\ \ \alpha_i, \beta_i \in \mathbb{N}. \tag{1.2}$$

Existence results for such problems can be found for instance in [6]. Here, we demonstrate the application and simplicity of computing the series solution of boundary value problem (BVP) of type (1.1) using the Parker-Sochacki Method. The Parker-Sochacki (PS) Method [7] is an iterative procedure for computing the coefficients of power series solution of first order initial value problems. Therefore by reducing problem (1.1) to a first order system, and computing all the missing initial data, the PS method is applied to generate the coefficients of the Maclaurin series expansion of the solution function through a recursive algorithm. The generated coefficients turned out to coincide with first Nth Picard iterates of the same problem. However, it is worth mentioning that the proposed method is devoid of known impracticality of the Picard iteration.

The PS method has received research attentions in the past years both for initial value problems and partial differential equations see, e.g, [8], [9], [10], [11]. In this paper, for the first time, the method is applied to a general nonlinear two-point boundary value problems. As an example of higher order problem, in Section 4, the method is successfully applied to the Falkner-Skan equation on finite domain.

2 The Parker-Sochacki Method

Here we demonstrate the PS method by considering a general first order system

$$y' = f(t, y),\ t \in (t_0, T],\ y(t_0) = y_0 \tag{2.1}$$

where $t_0, T \in \mathbb{R}$ are constants satisfying $t_0 \leq T$ and $f : (t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial function and $n \geq 1$. Without loss of generality, let us choose $t_0 = 0$. Note that this choice is not a restriction on the applicability of the method as the scaling $t \to t - t_0$ can always bring problems with $t_0 \neq 0$

to this case. The procedure to applying PS method to (2.1) goes thus: firstly we assume series solution for all the dependent variables

$$y = \sum_{i=0}^{N} y_i t^i, \quad y' = \sum_{i=0}^{N} y'_i t^i. \tag{2.2}$$

Next, differentiating and shifting the indexes appropriately, it holds

$$y' = \frac{dy}{dt}\left(\sum_{i=0}^{N} y_i t^i\right) = \sum_{i=0}^{N} i y_i t^{i-1} = \sum_{i=0}^{N} (i+1) y_{i+1} t^i.$$

Hence we obtain $\sum_{i=0}^{N} y'_i t^i = \sum_{i=0}^{N} (i+1) y_{i+1} t^i$, which in turn yields a recurrence relation for the coefficients

$$y_{i+1} = \frac{y'_i}{i+1} = \frac{f(t, y_i)}{i+1} \tag{2.3}$$

with $y_0 = y(0)$. Finally, using the above coefficients in (2.2), the truncated series solution

$$y(t) = \sum_{i=0}^{N} y_i t^i \tag{2.4}$$

is obtained.

2.1 Operations on power series

Based on the explicit form of $f(t, y_i)$ in (2.3), one often have to perform basic arithmetic operations on power series in order to successfully apply PS method. While addition and subtraction is straightforward [12], those of multiplication and division are not easy to come by. Let $x = \sum_{i\geq 0} x_i t^i$, $y = \sum_{i\geq 0} y_i t^i$. It has been established in [12] that the product (called the Cauchy product) of the two series is given by

$$p = \left(\sum_{i\geq 0} x_i t^i\right)\left(\sum_{i\geq 0} y_i t^i\right) = \sum_{i\geq 0} p_i t^i \tag{2.5}$$

where the coefficients p_i are recursively given by

$$p_i = \sum_{j=0}^{i} y_j x_{i-j}.$$

Integer powers of series can therefore be computed by repeated use of the Cauchy product above. The division formula for power series also follows from (2.5). If we define the series $w = \frac{x}{y}$ then (2.5) implies that

$$x_i = \sum_{j=0}^{i} y_j w_{i-j}.$$

Assuming that $y_0 \neq 0$, rearranging the terms gives

$$w_i = \frac{1}{y_0}\left(x_i - \sum_{j=0}^{i} w_j y_{i-j}\right).$$

For cases where division by zero arise, we refer to [13] for a smart way of circumventing this problem using the idea of 'shifting' the series until the first non-zero y_0.

2.2 Illustrative example

Example 2.1. *We consider the initial value problem*

$$y''(t) + 3y(t) - 2y(t)^3 = \cos t \sin 2t, \quad y(0) = 0, \;\; y'(0) = 1.$$

Employing the variable substitutions $u = y, v = y', w = \sin t, z = \cos t$, the problem reduces to a first order system

$$u' = v, v' = 2w - 2w^3 + 2u^3 - 3u, w' = z, z' = -w.$$

Now applying (2.3) *to each of the first order problem, we obtain the difference scheme*

$$u_{i+1} = \frac{v_i}{i+1}, \quad u_0 = 0 \tag{2.6}$$

$$v_{i+1} = \frac{2w_i - 2(w^3)_i + 2(u^3)_i - 3u_i}{i+1}, \quad v_0 = 1 \tag{2.7}$$

$$w_{i+1} = \frac{z_i}{i+1}, \quad w_0 = 0 \tag{2.8}$$

$$z_{i+1} = -\frac{w_i}{i+1}, \quad z_0 = 1 \tag{2.9}$$

whose first five iterates are computed to

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ w_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 0 \\ -1/6 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} u_4 \\ v_4 \\ w_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/24 \\ 0 \\ 1/24 \end{pmatrix}, \begin{pmatrix} u_5 \\ v_5 \\ w_5 \\ z_5 \end{pmatrix} = \begin{pmatrix} 1/120 \\ 0 \\ 1/120 \\ 0 \end{pmatrix}.$$

Hence, according to (2.4)*, the series solution is obtained (for $N = 19$) as*

$$\begin{aligned} y(t) = \sum_{i=0}^{N} u_i t^i &= t - \frac{1}{6}\, t^3 + \frac{1}{120}\, t^5 - \frac{1}{5040}\, t^7 + \frac{1}{362880}\, t^9 \\ &- \frac{1}{39916800}\, t^{11} + \frac{1}{6227020800}\, t^{13} - \frac{1}{1307674368000}\, t^{15} \\ &+ \frac{1}{355687428096000}\, t^{17} - \frac{1}{121645100408832000}\, t^{19} + O(t^{20}) \end{aligned}$$

which is in excellent agreement, term by term and up to that order, with the Maclaurin series of the exact solution $y(t) = \sin t$.

Remark 2.1. The applicability of the PS method to boundary value problems become limpid as soon as the missing initial conditions in the problem are computed. Several methods abound in the literature for computing missing initial data in boundary value problems. The most popular among these methods is the shooting method, which remains an important method as long as a convergent algorithm can be devised. Other known methods include the Lie Group method (or group transformation technique) [14, Chap. 9] and the Invariant Imbedding methods [15]. In the sequel, we shall briefly discuss the general shooting procedure for second order two-point boundary value problems with the assumption that the straightforward extension to higher order problems is clear.

3 Computation of Missing Initial Data

3.1 Shooting method

To illustrate the basic idea of shooting method, let us consider the general second-order nonlinear boundary value problem with Dirichlet boundary conditions of the form

$$y''(x) = f(x, y, y'), \quad a < x < b, \quad y(a) = \alpha, \quad y(b) = \beta. \tag{3.1}$$

The main idea of shooting method is to replace the above boundary value problem by a corresponding initial value problem

$$w''(x) = f(x, w, w'), \quad a < x < b, \quad w(a) = \alpha, \quad w'(a) = s \tag{3.2}$$

where $s = s_k$ is obtained iteratively in such a way that the other boundary condition holds via

$$\lim_{k\to\infty} w(b, s_k) = \beta.$$

In other words, the boundary condition at $y(b) = \beta$ is enforced by solving the nonlinear equation

$$g(s) = w(b, s) - \beta = 0. \tag{3.3}$$

The common nonlinear solvers for such problems are the Newton and Secant methods. Newton's method applied to (3.3) gives an iterative scheme

$$s_k = s_{k-1} - \frac{g(s_{k-1})}{\left(\frac{dg}{ds}\right)(b, s_{k-1})} = s_{k-1} - \frac{w(b, s_{k-1}) - \beta}{\left(\frac{\partial w}{\partial s}\right)(b, s_{k-1})}.$$

Note that the term $\frac{\partial w}{\partial s}(b, s_{k-1})$ in the above is not available since no explicit solution of (3.2) is available at this stage. Thus to compute $\frac{\partial w}{\partial s}(b, s_{k-1})$, we assume $w = w(x, s)$ in (3.2) where the variables x and s are assumed independent. It therefore holds

$$\begin{aligned}\frac{\partial}{\partial s} w''(x, s) &= \frac{\partial f}{\partial s}(x, w(x, s), w'(x, s)) \\ &= f_w(x, w(x, s), w'(x, s))\frac{\partial w}{\partial s} + f_{w'}(x, w(x, s), w'(x, s))\frac{\partial w'}{\partial s}.\end{aligned}$$

If we set the unknown slope $\frac{\partial w}{\partial s}(b, s_{k-1}) = z(x, s_{k-1})$ then $z(x, s_{k-1})$ is a solution of

$$z'' = f_w(x, w, w')z + f_{w'}(x, w, w')z' \tag{3.4}$$

with initial conditions $z(a) = 0, \quad z'(a) = 1$.

Altogether, the missing initial data $s = y'(a)$ is obtained by solving iteratively

$$w''(x) = f(x, w, w'), \quad a < x < b, \quad w(a) = \alpha, \quad w'(a) = s_{k-1}; \tag{3.5}$$

$$z'' = f_w(x, w, w')z + f_{w'}(x, w, w')z', \quad z(a) = 0, \quad z'(a) = 1; \tag{3.6}$$

$$s_k = s_{k-1} - \frac{w(b, s_{k-1}) - \beta}{\left(\frac{\partial w}{\partial s}\right)(b, s_{k-1})}. \tag{3.7}$$

The approach described above extends to higher order problems and cases of multiple missing data. However, we admit that dealing with the latter becomes more challenging. In the sequel, we describe a method that comes handy in the case of multiple missing initial conditions.

3.2 Root finding algorithm

Here the missing data are set to some constant parameters as in (3.2). The corresponding initial value problem is then solved using the proposed method as illustrated in Example 2.1. Finally, the obtained series solution is truncated and upon imposing the remaining boundary condition(s), the missing data are computed with the aid of root finding algorithm (e.g. fsolve in Maple) to obtain an approximation to the unknown initial data.

4 Examples

Example 4.1. *We consider the Dirichlet-Dirichlet BVP*

$$y'' = 0.5(1+x+y)^3, \quad 0 < x < 1,\ y(0) = y(1) = 0$$

with exact solution $y(x) = \frac{2}{2-x} - x - 1$.

Here, $f(x, w, w') = 0.5(1 + x + w)^3$. *So starting with* $s = 0$ *we solve*

$$w'' = 0.5(1+x+w)^3, \quad w(0) = 0,\ w'(0) = s_{k-1}, \tag{4.1}$$

$$v'' = 1.5v(1+x+w)^2, \quad v(0) = 1,\ v'(0) = 1, \tag{4.2}$$

$$s_k = s_{k-1} - \frac{w(1, s_{k-1})}{\left(\frac{\partial w}{\partial s}\right)(1, s_{k-1})} \tag{4.3}$$

iteratively to obtain $s = s_{12} = -0.5$. *We now solve the IVP*

$$y'' = 0.5(1+x+y)^3, \quad y(0) = 0,\ y'(0) = -0.5$$

using the proposed method. Therefore we set $u = y, v = y', t = 1 + x + y$ *to reduce the problem to a first order system*

$$u' = v, \quad v' = 0.5t^3, \quad t' = v$$

and consequently obtained the difference scheme

$$u_{i+1} = \frac{v_i}{i+1}, \quad u_0 = 0 \tag{4.4}$$

$$v_{i+1} = \frac{0.5(t^3)_i}{i+1}, \quad v_0 = -0.5 \tag{4.5}$$

$$t_{i+1} = \frac{v_i}{i+1}, \quad t_0 = 1, t_1 = 1 + v_0. \tag{4.6}$$

In the above, the term $(t^3)_i$ *is obtained through repeated use of Cauchy product as*

$$(t^3)_i = \sum_{j=0}^{i} \left(\sum_{k=0}^{j} t_k t_{j-k} \right) t_{i-j}.$$

The first few iterates of (4.4)-(4.6) *were computed as*

$$\begin{pmatrix} u_1 \\ v_1 \\ t_1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.375 \\ 0.25 \end{pmatrix}, \begin{pmatrix} u_3 \\ v_3 \\ t_3 \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0.25 \\ 0.125 \end{pmatrix}, \ldots$$

from which the series solution is obtained (for $N = 10$*) as*

$$y(x) = -0.5\,x + 0.25\,x^2 + 0.125\,x^3 + 0.0625\,x^4 + 0.03125\,x^5 \tag{4.7}$$

$$+ 0.015625\,x^6 + 0.0078125\,x^7 + 0.00390625\,x^8 \tag{4.8}$$

$$+ 0.001953125\,x^9 + 0.0009765625\,x^{10} + O(x^{11}). \tag{4.9}$$

Observe that the solution obtained is exact, up to that order N, when compared with Taylor expansion of the exact solution

$$\frac{2}{2-x} - x - 1 = -\frac{1}{2}\,x + \frac{1}{4}\,x^2 + \frac{1}{8}\,x^3 + \frac{1}{16}\,x^4 + \frac{1}{32}\,x^5 \tag{4.10}$$

$$+ \frac{1}{64}\,x^6 + \frac{1}{128}\,x^7 + \frac{1}{256}\,x^8 + \frac{1}{512}\,x^9 \tag{4.11}$$

$$+ \frac{1}{1024}\,x^{10} + O\left(x^{11}\right). \tag{4.12}$$

Example 4.2. *The nonlinear Dirichlet-Dirichlet BVP*

$$y'' = -\frac{(y')^2}{y}, \quad y(0) = 1,\ y(1) = 2$$

with exact solution $y(x) = \sqrt{3x+1}$.

Following (3.1), *we obtained on fourth iteration* $y'(0) = 1.5$. *Therefore, we solve the IVP*

$$y'' = -\frac{(y')^2}{y}, \quad y(0) = 1,\ y'(0) = 1.5$$

through the PS method by setting $v = y',\ \ w = 1/y$ *and solving the recursion*

$$y_{i+1} = \frac{v_i}{i+1}, \quad y_0 = 1, \tag{4.13}$$

$$v_{i+1} = -\frac{(v^2w)_i}{i+1}, \quad v_0 = 1.5 \tag{4.14}$$

$$w_{i+1} = -\frac{(w^2v)_i}{i+1}, \quad w_0 = 1/y_0 = 1. \tag{4.15}$$

For the coefficients $(wv^2)_i$ *and* $(w^2v)_i$ *above, repeated use of Cauchy product yields*

$$(wv^2)_i = \sum_{j=0}^{i}\left(\sum_{k=0}^{j} v_k v_{j-k}\right) w_{i-j}, \quad (w^2v)_i = \sum_{j=0}^{i}\left(\sum_{k=0}^{j} w_k w_{j-k}\right) v_{i-j} \tag{4.16}$$

which are then used in (4.13)-(4.15) *to obtain*

$$\begin{pmatrix} y_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -2.25 \\ -1.5 \end{pmatrix}, \begin{pmatrix} y_2 \\ v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1.125 \\ 5.0625 \\ 3.375 \end{pmatrix},$$

$$\begin{pmatrix} y_3 \\ v_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1.6875 \\ -12.65625 \\ -8.4375 \end{pmatrix}, \begin{pmatrix} y_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} -3.1640625 \\ 33.22265625 \\ 22.1484375 \end{pmatrix}, \ldots$$

Subsequently, the series solution to the BVP is

$$y(x) = 1 + 1.5\,x - 1.125\,x^2 + 1.6875\,x^3 - 3.1640625\,x^4 \tag{4.17}$$

$$+ 6.64453125\,x^5 - 14.95019531\,x^6 + 35.23974610\,x^7 \tag{4.18}$$

$$- 85.89688111\,x^8 + 214.7422028\,x^9 - 547.5926171\,x^{10} \tag{4.19}$$

$$+ 1418.762690\,x^{11} - 3724.252061\,x^{12} \tag{4.20}$$

$$+ 9883.592008\,x^{13} + O(x^{14}) \tag{4.21}$$

which agrees well with Taylor expansion of the exact solution

$$\sqrt{3x+1} = 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3 - \frac{405}{128}x^4 + \frac{1701}{256}x^5 \tag{4.22}$$
$$- \frac{15309}{1024}x^6 + \frac{72171}{2048}x^7 - \frac{2814669}{32768}x^8 + \frac{14073345}{65536}x^9 \tag{4.23}$$
$$- \frac{143548119}{262144}x^{10} + \frac{743840253}{524288}x^{11} - \frac{15620645313}{4194304}x^{12} \tag{4.24}$$
$$+ \frac{82909578969}{8388608}x^{13} + O\left(x^{14}\right). \tag{4.25}$$

Example 4.3. *We consider the problem with Neumman-Neumann boundary conditions*

$$y'' = -e^{-2y},\ 0 < x < 1,\ y'(0) = 1,\ y'(1) = \frac{1}{2}$$

whose exact solution is known to be $y(x) = \ln(1+x)$.

The missing parameter $s = y(0)$ *is obtained on first iteration of shooting algorithm as* $s = y(0) = -2.8 \times 10^{-9}$. *Hence it suffices to solve the IVP*

$$y'' = -e^{-2y},\ 0 < x < 1,\ y(0) = -2.8 \times 10^{-9},\ y'(0) = 1$$

with the PS method. Employing the variable substitution $u = y,\ v = y',\ w = e^{-2y}$ *the problem reduces to*

$$u' = v, \quad v' = -w, \quad w' = -2wv \tag{4.26}$$

whose PS solution is obtained iteratively through

$$u_{i+1} = \frac{v_i}{i+1}, \quad u_0 = -2.8 \times 10^{-9} \tag{4.27}$$
$$v_{i+1} = -\frac{w_i}{i+1}, \quad v_0 = 1 \tag{4.28}$$
$$w_{i+1} = -2\frac{\sum_{j=0}^{i} v_j w_{i-j}}{i+1}, \quad w_0 = e^{-2u_0}. \tag{4.29}$$

We obtained

$$u_1 = 1,\ u_2 = -\frac{1}{2},\ u_3 = \frac{1}{3},\ u_4 = -\frac{1}{4},\ u_5 = \frac{1}{5},\ \ldots, u_i = (-1)^{i+1}\frac{1}{i}.$$

Hence, in line with the PS method, the series solution is given by (choosing $N = \infty$*)*

$$y(x) = \sum_{i=0}^{\infty} u_i x^i = \sum_{i=0}^{\infty} (-1)^{i+1}\frac{x^i}{i} = \ln(1+x).$$

Example 4.4 (The Falkner-Skan Equation)**.** *Consider the boundary layer flow problem given by the third order BVP*

$$f''' + ff'' + \beta(1 - f'^2) = 0, \quad f(0) = 0, f'(0) = 0, \lim_{\eta\to\infty} f'(\eta) = 1. \tag{4.30}$$

Let us set the missing data $s = f''(0)$ *so that we consider the IVP*

$$f''' + ff'' + \beta(1 - f'^2), \quad f(0) = 0,\ \ f'(0) = 0,\ \ f''(0) = s. \tag{4.31}$$

The IVP (4.31) is now solved using the PS method. Adopting the variable substitution $u = f', z = f''$, problem (4.31) reduces to the system

$$f' = u, \quad f(0) = 0, \tag{4.32}$$
$$u' = z, \quad u(0) = 0, \tag{4.33}$$
$$z' = -fz + \beta u^2 - \beta, \quad z(0) = s. \tag{4.34}$$

The PS solution is thus obtained via the difference equations

$$f_{i+1} = \frac{u_i}{i+1}, \quad f_0 = 0, \tag{4.35}$$
$$u_{i+1} = \frac{z_i}{i+1}, \quad u_0 = 0, \tag{4.36}$$
$$z_{i+1} = \frac{-(fz)_i + \beta(u^2)_i}{i+1}, \quad z_0 = s, \;\; z_1 = -f_0 z_0 + \beta u_0^2 - \beta. \tag{4.37}$$

The first few iterates are computed as

$$f_1 = 0, \; f_2 = \frac{s}{2}, \; f_3 = -\frac{\beta}{6}, \; f_4 = 0, \; f_5 = s^2\left(\frac{\beta}{60} - \frac{1}{120}\right).$$

The series solution is therefore given by

$$f(\eta) = \frac{1}{2}\, s\eta^2 - \frac{1}{6}\,\beta\,\eta^3 + \left(\frac{\beta}{60} - \frac{1}{120}\right) s^2\eta^5 + \left(\frac{1}{180}s\beta - \frac{1}{120}\beta^2 s\right)\eta^6 + \ldots$$

For $\beta = 2$, and starting with $s_0 = 1.5$, we obtained on seventh iteration of shooting algorithm $s = f''(0) = 1.6872$ which agrees perfectly with the famous numerical results of [16]. Furthermore, Table 1 shows the obtained numerical results for $f'(\eta)$ and $f''(\eta)$. The behaviors of $f'(\eta)$ and $f''(\eta)$ are in excellent agreement with the theory by [17], namely, that $f'(\eta)$ increases with η and $f''(\eta)$ decays to zero as $\eta \to \infty$. In the presented results, our truncated (or free) boundary is taken as $\eta_\infty \approx 2$.

Example 4.5. *Consider the fifth order BVP*

$$u^{(5)}(x) = e^{-x}u^2(x), \quad 0 < x < 1$$

subject to

$$u(0) = u'(0) = u''(0) = 1, \quad u(1) = u'(1) = e$$

with exact solution $u(x) = e^x$.

Suppose we can write $u(x) = \sum_{i \geq 0} u_i x^i$ and and let us denote the missing initial data by $u'''(0) = a, \;\; u^{(4)}(0) = b$. The constants a and b are to be determined from the boundary conditions $u(1) = u'(1) = e$. Adopting the variable substitutions $v = u', \; w = u'', \; t = u''', \; z = u^{(4)}, \; p = e^{-x}$ the BVP reduces to a first order system

$$u' = v; \; v' = w; \; w' = t; \; t' = z; \; z' = pu^2; \; p' = -p.$$

Then the coefficients u_i are obtained recursively through

$$u_{i+1} = \frac{v_i}{i+1}; \; v_{i+1} = \frac{w_i}{i+1}; \; w_{i+1} = \frac{t_i}{i+1}; \; t_{i+1} = \frac{z_i}{i+1}; \; z_{i+1} = \frac{(pu^2)_i}{i+1}; \; p_{i+1} = -\frac{p_i}{i+1}$$

with $u_0 = v_0 = w_0 = 1, t_0 = a, z_0 = b$.

We computed

$$u_1 = 1, \; u_2 = \frac{1}{2}, \; u_3 = \frac{a}{6}, \; u_4 = \frac{b}{24}, \; u_5 = \frac{1}{120}, \; u_6 = \frac{1}{720}, \; u_7 = \frac{1}{5040}, \; u_8 = \left(\frac{a}{20160} - \frac{1}{40320}\right)$$

and so on. Hence the desired series solution is

$$u(x) = 1+x+\frac{x^2}{2}+\frac{ax^3}{6}+\frac{bx^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+\left(\frac{a}{20160}-\frac{1}{40320}\right)x^8+\left(\frac{b}{181440}-\frac{1}{362880}\right)x^9+\ldots \tag{4.38}$$

To compute the unknown parameters a and b, we truncate the series solution at $N = 15$. Upon imposing the boundary conditions $u(1) = u'(1) = e$, we obtain through the fsolve command in Maple

$$a = 0.9999999824, \quad b = 1.000000049.$$

Hence, the required series solution is

$$\begin{aligned} u(x) = 1 + x + \frac{1}{2}x^2 + 0.1666666637\,x^3 + 0.04166666871\,x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 \\ + 0.00002480158643\,x^8 + 0.000002755732193\,x^9 + \frac{1}{3628800}x^{10} + 0.000000025052108\,x^{11} \\ + 0.0000000020876756\,x^{12} + 0.00000000016059047\,x^{13} + 1.147072\times 10^{-11}\,x^{14} \\ + 7.64718\times 10^{-13}\,x^{15} + O(x^{16}) \end{aligned}$$

which is an excellent approximation of the exact solution e^x.

Table 1. Numerical results for $f'(\eta)$ and $f''(\eta)$ by the proposed method

η	$f'(\eta)$	$f''(\eta)$
0.2	0.29794	1.29688
0.4	0.52189	0.95270
0.6	0.68342	0.67383
0.8	0.79586	0.46106
1.0	0.87170	0.30609
1.2	0.92138	0.19748
1.4	0.95303	0.12391
1.6	0.97263	0.07561
1.8	0.98444	0.04485
2.0	0.99135	0.02582

5 Conclusions

We have introduced a new analytic technique for solving nonlinear boundary value problems. In addition to the new technique being easy-to-implement, application to various examples in Section 4 revealed that it is also highly accurate. Therefore the proposed method remains a viable alternative to notable approximation techniques in the literature including the homotopy perturbation method, variational iteration method and adomian decomposition method among others.

Competing Interests

Authors have declared that no competing interests exist.

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