Journal of Advances in Mathematics and Computer Science

23(2): 1-22, 2017; Article no.JAMCS.34336

Previously known as British Journal of Mathematics & Computer Science

ISSN: 2231-0851

Convergence Analysis of a Non-overlapping DDM for Optimal Absorbing Boundary Control Problems Governed by Wave Equations

Keying Ma[1*] and Wenyue Liu[1]

[1] *School of Mathematics, Shandong University, Jinan 250100, P.R. China.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2017/34336

Editor(s):

(1) Nikolaos Dimitriou Bagis, Department of Informatics and Mathematics, Aristotelian University of Thessaloniki, Greece.

Reviewers:

(1) Abdullah Sonmezoglu, Bozok University, Yozgat, Turkey.

(2) Sie Long Kek, Universiti Tun Hussein Onn Malaysia, Malaysia.

(3) Giuliani Donatella, University in Bologna, Italy.

Complete Peer review History: http://www.sciencedomain.org/review-history/19835

Received: 24th May 2017

Accepted: 23rd June 2017

Published: 3rd July 2017

Original Research Article

Abstract

A non-overlapping domain decomposition method (DDM) is described to solve optimal boundary control problems governed by wave equations with absorbing boundary condition. The whole domain is divided into non-overlapping subdomains, and the global optimal boundary control problem is decomposed into local problems in these subdomains. An integral mean method is utilized to present an explicit flux calculation on the inter-domain boundary in order to communicate the local problems on the interfaces between subdomains. We establish the full parallel and discrete schemes for solving these local problems, and prove the stability of the schemes. A priori error estimates in suitable natural norms are derived for the state, co-state and control variables.

Keywords: Wave equations; optimal absorbing boundary control problems; non-overlapping DDM; integral mean method; error estimates.

2010 Mathematics Subject Classification: 65N12, 65N15, 65N22.

**Corresponding author: E-mail: makeying@sdu.edu.cn;*

1 Introduction

Optimal control problems governed by wave equations are widely used in many fields, such as in medical science [1], seismic wave [2] and acoustic wave [3]. Generally speaking, these optimal control problems aim to find a control variable which makes the state variable tend to an expected target state in the process of optimizing (maximize/minimize) the objective functional and meanwhile, the state and control variables are subjected to wave equations. [4] and [5] considered some numerical methods for optimal control problems governed by wave equations.

As well known, for the problem on large domain, the traditional numerical methods, such as finite element methods, always produce large amounts of calculation which can be settled efficiently by the method of parallel computations. A natural way in parallel computations is the non-overlapping domain decomposition method (DDM). [6][7][8][9][10][11] discussed the applications of non-overlapping DDM for optimal boundary control problems governed by partial differential equations (PDEs). This method can divide the whole domain into many subdomains, and decompose the global problem into many local problems, which are independent in the subdomains and can be calculated parallel. Hence, this method can reduce much amounts of calculation. An important character of this method is how to build inter-domain boundary conditions of state/co-state variables between subdomains. For an example, for optimal boundary control problems governed by hyperbolic equations, [11] presented an iterative non-overlapping DDM, utilized Robin condition as the inter-domain boundary condition and proved the convergence of the method.

When calculating solutions to the problem in an unbounded domain, it is often essential to introduce artificial boundaries to limit the area of computation. Important fields of applying artificial boundaries are local weather prediction [12][13], geophysical calculation involving acoustic waves [14] or elastic waves [15]. For wave equations, Engquist et al. [16] developed the theory of absorbing boundary condition, which is a kind of artificial boundaries and consists of the time and space derivatives of the function. This boundary condition not only guarantees stable difference approximations but also minimizes the (unphysical) artificial reflections which occur at the boundaries. Cowsar, Dupont and Wheeler [17] proposed the mixed finite element method for linear hyperbolic equation with absorbing boundary condition. In [18], Bamberger at al. discussed a DDM for the acoustic wave equation with absorbing boundary condition.

Lagnese et al. [19] studied a time-domain decomposition method for an optimal boundary control problem governed by wave equations with absorbing boundary condition. The objective functional consisted of the final time values of both the function and the time derivation of the function. They decomposed the global optimal control problem into local problems on each time interval, built inter-domain boundary conditions for state/co-state variables between subdomains, and proved the convergence of the method.

The purpose of this paper is to present another type of non-overlapping DDM for the model problem in [19]. Based on our former work [20] and different from the methods in [6][7][8][9][10][11], we utilize an integral mean method to present an explicit flux calculation on the inter-domain boundary and establish the non-overlapping domain decomposition scheme. This type of non-overlapping DDM has been presented for parabolic equation in [20][21][22][23], wave equation [24] and convection-diffusion equation [25]. Nevertheless, we did not extend this method to optimal boundary control problems governed by wave equations, especially with absorbing boundary condition. This paper is one of our sequent research papers. To our best knowledge, there is no similar work on this topic.

An outline of this paper is as follows. In §2, we introduce the optimal boundary control problem governed by wave equations with absorbing boundary condition, and deduce the co-state equation and the optimality condition. In §3, we recall the non-overlapping DDM by using the integral mean method in [20]. Then, we utilize this method to establish the full discrete schemes, and prove the stability of the schemes. In §4, we derive a priori error estimates in suitable natural norms for the state, co-state and control variables. Finally, we draw the conclusions in §5.

2 Optimal Boundary Control Problem

Let $\Omega \subset \mathbf{R^2}$ be a bounded convex domain with smooth boundary $\partial\Omega$ and $[0, T]$ be a time interval. Let $\partial\Omega = \Gamma_N \cup \Gamma_D$, $\Gamma_N \cap \Gamma_D = \varnothing, \Gamma_N$ and Γ_D be Neumann and Dirichlet type boundary, respectively.

Throughout the paper, the standard notations [26] are used for the Lebesgue space $L^m(\Omega)$, $1 \leq m \leq \infty$ and the Sobolev space $H^s(\Omega)$, $0 \leq s \leq \infty$ with the associated norms $\|\cdot\|_s$ and seminorms $|\cdot|_s$. We will assume C to be a generic positive constant independent of mesh size h (to be defined in the next section), but may depend on the size of Ω and can take different values at different places.

Denote the state space by $W = L^2(0, T; V)$ with $V = H^1(\Omega)$ and the control space by $X = L^2(0, T; M)$ with $M \subseteq L^2(\Gamma_N)$. We consider the following optimal boundary control problem governed by wave equations with absorbing boundary condition [19]:

$$\min_{u\in X} J(u) = \min_{u\in X}\Big\{\frac{\gamma}{2}\Big\{\int_\Omega |y(T) - z_0|^2 + \int_\Omega |\frac{\partial y}{\partial t}(T) - z_1|^2\Big\} + \frac{\alpha}{2}\int_{\Gamma_N\times(0,T)} |u|^2\Big\} \tag{2.1}$$

where, the state variable $y \in W$ and control variable $u \in X$ satisfy

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} - \Delta y(x,t) + cy = f(x,t), & \text{in } \Omega \times (0,T), \\ \dfrac{\partial y(x,t)}{\partial \vec{\nu}} + \dfrac{\partial y}{\partial t} = u(x,t), & \text{on } \Gamma_N \times (0,T), \\ y(x,t) = 0, & \text{on } \Gamma_D \times (0,T), \\ y(x,0) = y_1(x), & \text{in } \Omega, \\ \dfrac{\partial y}{\partial t}(x,0) = y_2(x), & \text{in } \Omega. \end{cases} \tag{2.2}$$

In equations (2.1)-(2.2), $\vec{\nu}$ is the unit outward normal vector on Γ_N, $f(x,t), y_1(x)$ and $y_2(x)$ are known functions, $z_0(x)$, $z_1(x)$ are known target functions, γ and α are positive given wight coefficients. The boundary condition on Γ_N is called as absorbing boundary condition, which includes the time and space derivatives of the state y. Here, we consider the objective functional $J(u)$ with final time values of both y and $\partial y/\partial t$. The overall idea of equation (2.1) is to drive y and $\partial y/\partial t$ as close as possible to the target state $z_0(x)$ and $z_1(x)$ respectively, while the second term penalizes excessive control cost.

According to the optimal control theory in [27][28][29] and [19], we can obtain the adjoint equation of

$$\begin{cases} \dfrac{\partial^2 p(x,t)}{\partial t^2} - \Delta p(x,t) + cp = 0, & \text{in } \Omega \times (0,T), \\ \dfrac{\partial p(x,t)}{\partial \vec{\nu}} - d_t p = 0, & \text{on } \Gamma_N \times (0,T), \\ p(x,t) = 0, & \text{on } \Gamma_D \times (0,T), \\ p(x,T) = \gamma(\dfrac{\partial y}{\partial t}(T) - z_1), & \text{in } \Omega, \\ \dfrac{\partial p}{\partial t}(x,T) = -\gamma(y(T) - z_0), & \text{in } \Omega. \end{cases} \tag{2.3}$$

where, $p \in W$ is called the co-state variable of y. And, $d_t:\ L^2(0,T;M) \longmapsto (H^1(0,T;M))^*$ is a bounded linear operator satisfying [19]

$$< d_t p, \phi >_{\Gamma_N\times(0,T)} := -\int_0^T (p, \frac{d\phi}{dt})_M dt,\ \forall\, \phi \in H^1(0,T;M).$$

where, space $(H^1(0,T;M))^*$ is the dual space of space $H^1(0,T;M)$, $<\cdot,\cdot>_{\Gamma_N\times(0,T)}$ denotes the inner product in the $\big(H^1(0,T;M))^*, H^1(0,T;M)\big)$ duality pairing. We note that $d_t p$ is not the time derivative dp/dt of p in the sense of distributions.

We know that when the objective functional J reaches its optimum, the control variable $u \in X$ should satisfy ([27][28][29])

$$J'(u)(\bar{u}-u) = \int_{\Gamma_N\times(0,T)} (\alpha u + p|_{\Gamma_N})(\tilde{u}-u) \geq 0,\ \forall\ \tilde{u}\ \in\ X. \tag{2.4}$$

This inequality is called as the optimality condition.

Then, the optimal boundary control problem (2.1)-(2.2) is equivalent to an optimality system, which consists of the state equation (2.2), the co-state equation (2.3) and the optimality condition (2.4). We can get the solutions of equations (2.1)-(2.2) by solving the optimality system (2.2)-(2.4).

3 Non-Overlapping DDM

3.1 Approximation schemes

To avoid large amounts of computational work by the traditional numerical methods to the optimality system (2.2)-(2.4), we will build a non-overlapping domain decomposition scheme.

For simplicity and without losing generality, we only discuss the case of two subdomains. But the algorithms and theories can be extended to the case of many subdomains. Divide Ω into two non-overlapping subdomains $\Omega_i (i = 1,2)$ by an inter-domain boundary Γ (see Fig. 1)

$$\Omega = \Omega_1 \cup \Omega_2,\ \ \Omega_1 \cap \Omega_2 = \Gamma.$$

Let $\Gamma_{D,i} = \Gamma_D \cap \partial\Omega_i$ and $\Gamma_{N,i} = \Gamma_N \cap \partial\Omega_i$, $\Gamma_{D,i} \neq \emptyset$, $\Gamma_{N,i} \neq \emptyset$. Define $\vec{\nu}_\Gamma$ to be the unit normal vector on Γ, which points from Ω_1 toward Ω_2. We suppose that this decomposition guarantee the global and local problems hold enough regularities.

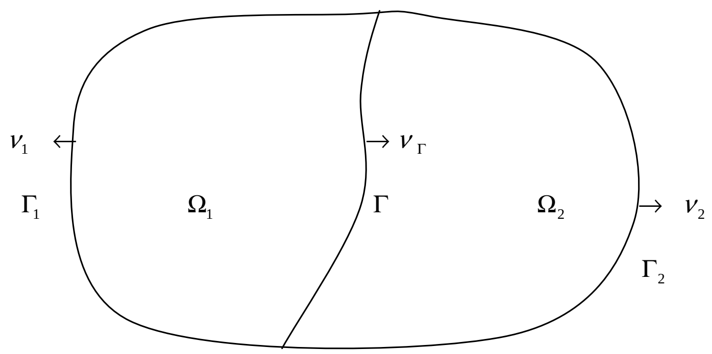

Fig. 1. Subdomains Ω_i and inter-domain boundary Γ

Let T_i^h be a quasi-uniform partition of subdomains $\Omega_i (i = 1, 2)$ and $T^h = T_1^h \cup T_2^h$. Define $\bar{\Omega} = \cup_{\tau \in T^h} \bar{\tau}$. Here, h denotes the maximal diameter of element $\tau \in \mathcal{T}^h$. For two neighboring elements $\tau, \bar{\tau} \in T^h$, they have either only one common vertex or edge. Let $V^h \subset V$ is a finite element space satisfying

$$V^h = \{v \in H^1(\Omega) : \ v|_\tau \in \mathcal{P}_1(\tau), \ \forall \ \tau \in T^h\},$$

where $\mathcal{P}_1(\tau)$ denotes the polynomials of degree less than or equal to 1 on τ. Denote $W^h = L^2(0, T; V^h)$

Similarly, let $T_{U,i}^h$ be a quasi-uniform partition of $\Gamma_{N,i}$ and $T_U^h = T_{U,1}^h \cup T_{U,2}^h$. Let h_U denote the maximal diameter of element $\tau_U \in T_U^h$. For two neighboring elements $\tau_U, \tau_U' \in T_U^h$, they have only one common vertex. Let $M^h \subset M$ is a finite element space satisfying

$$M^h = \{v \in L^2(\Gamma_N) : \ v|_{\tau_U} \in \mathcal{P}_0(\tau), \ \forall \ \tau_U \in T_U^h\}.$$

where $\mathcal{P}_0(\tau_U)$ denotes the polynomials of degree 0 (i.e., constant) on τ_U. Denote $X^h = L^2(0, T; M^h)$.

From definitions above, we note that functions v in W^h have a well-defined jump $[v]$ on Γ:

$$[v](x) = \lim_{\lambda \to 0^+} v(x + \lambda \vec{\nu}_\Gamma) - \lim_{\lambda \to 0^-} v(x + \lambda \vec{\nu}_\Gamma). \tag{3.1}$$

To construct the scheme, for a small given constant $0 < H < \min\{diameter(\Omega_1), diameter(\Omega_2)\}$, we introduce an integral mean value of a given function $v \in H^1(\Omega)$ on Γ as ([20])

$$\overline{v}_H = \frac{1}{2H} \int_{-H}^{H} v(x + \lambda \vec{\nu}_\Gamma) d\lambda, \ \ \forall \ x \text{ on } \Gamma. \tag{3.2}$$

Generally, near the intersection of the boundary $\partial\Omega$ and inner boundary Γ, the value of v outside Ω may be needed to calculate the integral mean value $\overline{v}_H$ in (3.2). For a given function $v \in L^2(\Omega)$, we define ([20])

$$Ev(x) = \begin{cases} v(x), & x \in \Omega, \\ v(\tilde{x}), & x \notin \Omega, \end{cases} \tag{3.3}$$

where $\tilde{x} \in \Omega$ denotes the symmetric point of $x \notin \Omega$ with respect to $\partial\Omega$. By (3.3), we know $\overline{v}_H$ has the value on a strip domain$G = \{y | y = x + \lambda \vec{\nu}_\Gamma, \lambda \in [-H, H], \, x \, \text{on} \, \Gamma\}$, see Fig. 2.

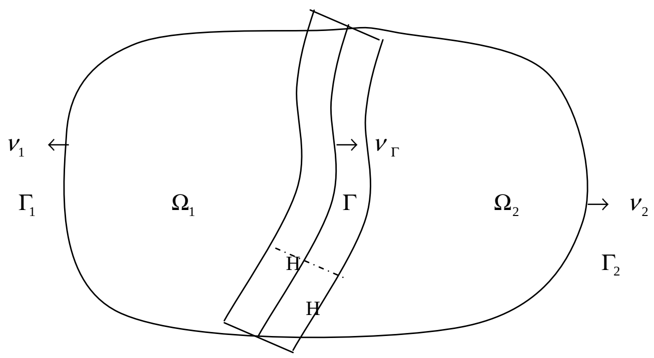

Fig. 2. The strip domain G with width $2H$

Let Δt be time step size, $N = T/\Delta t$, $t^n = n\Delta t, n = 1, \cdots, N$. For a given function v, we adopt the following notations. Set

$$\begin{aligned}
&v^n = v(t^n), && v^{n+\frac{1}{2}} = \frac{v^{n+1}+v^n}{2},\\
&v^{n,\theta} = \theta v^{n+1} + (1-2\theta)v^n + \theta v^{n-1}, && \theta \in (0,1),\\
&\partial_t v^n = \frac{v^{n+1}-v^n}{\Delta t}, && \partial_t v^{n-1} = \frac{v^n - v^{n-1}}{\Delta t},\\
&\delta v^n = \frac{v^{n+1}-v^{n-1}}{2\Delta t}, && \partial_t^2 v^n = \frac{v^{n+1}-2v^n+v^{n-1}}{(\Delta t)^2}.
\end{aligned} \tag{3.4}$$

By using of the integral mean non-overlapping DDM scheme in [20], we can define the full discrete schemes for the optimality system (2.2)-(2.4): Find the approximation solution $\{Y^n,\, U^n\}_{n=1}^N \in V^h \times M^h$ satisfying

$$\begin{aligned}
J(U) = &\sum_{n=1}^{N-1} \frac{\alpha}{2}\int_{\Gamma_N} |U^{n,\frac{1}{4}}|^2 \Delta t + \frac{\gamma}{2}\{\int_\Omega |Y^N - z_0|^2 + \int_\Omega |\partial_t Y^{N-1} - z_1|^2\}\\
&-\frac{\gamma}{4}(\Delta t)^2\{ <[\partial_t Y^{N-1} - z_1], \overline{(\partial_t Y^{N-1} - z_1)_{\vec{\nu}_{\Gamma,H}}} >_\Gamma + <[Y^N - z_0], \overline{(Y^N - z_0)_{\vec{\nu}_{\Gamma,H}}} >_\Gamma\\
&+\frac{KH^{-1}}{2}\{<[Y^N - z_0],[Y^N - z_0]>_\Gamma + <[\partial_t Y^{N-1} - z_1],[\partial_t Y^{N-1} - z_1]>_\Gamma\}\},
\end{aligned} \tag{3.5}$$

and

$$\begin{cases}
(\partial_t^2 Y^n, v) + (\nabla Y^{n,\frac{1}{4}}, \nabla v) + c(Y^{n,\frac{1}{4}}, v) + <\overline{Y^n_{\vec{\nu}_{\Gamma,H}}}, [v]>_\Gamma\\
\quad + <\overline{v_{\vec{\nu}_{\Gamma,H}}}, [Y^n]>_\Gamma + KH^{-1} <[Y^n],[v]>_\Gamma\\
= (f^{n,\frac{1}{4}}, v) + <U^{n,\frac{1}{4}}, v>_{\Gamma_N} - <\delta Y^n, v>_{\Gamma_N}, & \text{in } \Omega,\\
Y^0 = y_1(x), & \text{in } \Omega,\\
\partial_t Y^0 = y_2(x), & \text{in } \Omega,
\end{cases} \tag{3.6}$$

where $\forall\, v \in V_0^h = \{v \in V^h, v|_{\Gamma_D} = 0\}$;

$$\chi^n_{\vec{\nu}_\Gamma} = (\nabla E \chi^n)\cdot \vec{\nu}_\Gamma, \quad \text{for } \chi = Y,\, v; \quad K = \begin{cases} 1, & if\, G \subset \Omega,\\ 2, & if\, G \not\subset \Omega.\end{cases}$$

The notation $<\cdot,\cdot>_\Gamma$ (resp. $<\cdot,\cdot>_{\Gamma_N}$) is denoted as the L^2 inner product on the boundary Γ (resp. Γ_N). On Γ_N, δY^n is used to approximate $\partial y/\partial t$ in order to keep second order convergence rate for Δt.

Remark 3.1. In the scheme (3.6), the flux on Γ is computed explicitly from Y^n, so that Y^{n+1} can be computed on Ω_1 and Ω_2 fully parallel once Y^n, Y^{n-1} have been got. From the convergence analysis in Section 4, we will see that this scheme has good approximations.

From the optimal control theory in [27][28][29], we can deduce the full discrete schemes for the adjoint equation (2.3): Find $P^n \in V^h$ satisfying

$$\begin{cases}
(\partial_t^2 P^n, w) + (\nabla P^{n,\frac{1}{4}}, \nabla w) + c(P^{n,\frac{1}{4}}, w) + <\overline{P^n_{\vec{\nu}_{\Gamma,H}}}, [w]>_\Gamma\\
\quad + <\overline{w_{\vec{\nu}_{\Gamma,H}}}, [P^n]>_\Gamma + KH^{-1} <[P^n],[w]>_\Gamma = <\delta P^n, w>_{\Gamma_N}, & \text{in } \Omega,\\
P^N = \gamma(\partial_t Y^{N-1} - z_1), & \text{in } \Omega,\\
\partial_t P^{N-1} = -\gamma(Y^N - z_0), & \text{in } \Omega,
\end{cases} \tag{3.7}$$

for $\forall\, w \,\in\, V_0^h$ and for the optimality condition

$$< \alpha U^{n,\frac{1}{4}} + P^{n,\frac{1}{4}}, \tilde{u} - U^{n,\frac{1}{4}} >_{\Gamma_N} \;\geq\; 0, \quad \forall\; \tilde{u} \,\in\, M^h. \tag{3.8}$$

Hence, we establish the full discrete schemes (3.6)-(3.8) for the optimality system (2.2)-(2.4).

3.2 Stability of approximation schemes

To prove the stability of the full discrete schemes (3.6)-(3.8), we need the following notations and lemmas. For functions $\psi \,\in\, H^1(\Omega_1) \cup H^1(\Omega_2)$, we define [20]

$$|||\psi|||^2 = (\nabla\psi, \nabla\psi) + KH^{-1} < [\psi], [\psi] >_\Gamma, \tag{3.9}$$

and a bilinear form

$$b(\psi, \psi) = (\nabla\psi, \nabla\psi) + 2 < \overline{\psi_{\vec{\nu}_{\Gamma,H}}}, [\psi] >_\Gamma + KH^{-1} < [\psi], [\psi] >_\Gamma . \tag{3.10}$$

Lemma 3.1. *[20] There exists a positive constant $C_0 = 1 - \dfrac{\sqrt{2}}{2}$ such that for constant $H > 0$*

$$b(\psi, \psi) \;\geq\; C_0|||\psi|||^2, \;\; \forall\, \psi \,\in\, V^h. \tag{3.11}$$

Define an "energy" norm ([24])

$$\begin{aligned} \|\psi^n\|^2_{\mathcal{E}_1} \;\;&=\;\; \|\partial_t\psi^n\|^2 + b(\psi^{n+\frac{1}{2}}, \psi^{n+\frac{1}{2}}) + c\|\psi^{n+\frac{1}{2}}\|^2 \\ &\quad - \tfrac{(\Delta t)^2}{4}\big\{2 < \overline{\partial_t\psi^n_{\vec{\nu}_{\Gamma,H}}}, [\partial_t\psi^n] >_\Gamma + KH^{-1} < [\partial_t\psi^n], [\partial_t\psi^n] >_\Gamma\big\}. \end{aligned} \tag{3.12}$$

We turn to prove that the "energy" norm (3.12) is nonnegative under a time step constraint. To this end, we need the following inverse estimate ([30])

$$\|\nabla\psi\| \;\leq\; C_2 h^{-1}\|\psi\|, \tag{3.13}$$

and the trace inequality

$$\|\psi\|^2_{L^2(\Gamma)} \;\leq\; C_3 h^{-1}\|\psi\|^2. \tag{3.14}$$

Lemma 3.2. *[24] Denote L = H/h. There exists a positive constant C_1 such that for constant $H > 0$,*

$$\|\psi\|^2_{\mathcal{E}_1} \;\geq\; \frac{1}{2}\|\partial_t\psi^n\|^2 + b(\psi^{n+\frac{1}{2}}, \psi^{n+\frac{1}{2}}) + c\|\psi^{n+\frac{1}{2}}\|^2, \tag{3.15}$$

provided that $\Delta t \;\leq\; C_1 H$, where $C_1 = \sqrt{\dfrac{2}{C_2^2 L^2 + 3KC_3 L}}$.

Lemma 3.3. *If $f = 0$, $U = 0$, there exists a positive constant C such that for $n = 1, 2, \cdots, N$, the full discrete schemes* (3.6) *and* (3.7) *hold the following estimates in "energy" norm*

$$\|Y^n\|^2_{\mathcal{E}_1} \;\leq\; \|Y^{n-1}\|^2_{\mathcal{E}_1}, \tag{3.16}$$

$$\|P^{n-1}\|^2_{\mathcal{E}_1} \;\leq\; \|P^n\|^2_{\mathcal{E}_1}. \tag{3.17}$$

Proof. Estimate (3.16) can be obtained similarly from the proof of Lemma 3.5 in [24]. In equation (3.6), we take $v = \delta Y^n$ to get that

$$\begin{aligned}
&(\partial_t^2 Y^n, \delta Y^n) + (\nabla Y^{n,\frac14}, \nabla\delta Y^n) + c(Y^{n,\frac14}, \delta Y^n) + < \overline{Y^{n,\frac14}_{\vec{\nu}_{\Gamma,H}}}, [\delta Y^n] >_\Gamma \\
&\quad + < \overline{\delta Y^n_{\vec{\nu}_{\Gamma,H}}}, [Y^{n,\frac14}] >_\Gamma + KH^{-1} < [Y^{n,\frac14}], [\delta Y^n] >_\Gamma + < \delta Y^n, \delta Y^n >_{\Gamma_N} \\
&= < \overline{Y^{n,\frac14}_{\vec{\nu}_{\Gamma,H}}} - \overline{Y^n_{\vec{\nu}_{\Gamma,H}}}, [\delta Y^n] >_\Gamma + < \overline{\delta Y^n_{\vec{\nu}_{\Gamma,H}}}, [Y^{n,\frac14} - Y^n] >_\Gamma + KH^{-1} < [Y^{n,\frac14} - Y^n], [\delta Y^n] >_\Gamma .
\end{aligned} \tag{3.18}$$

By the notations (3.4), we have the following results.
(i)

$$(\partial_t^2 Y^n, \delta Y^n) = \frac{1}{2\Delta t}\{\|\partial_t Y^n\|^2 - \|\partial_t Y^{n-1}\|^2\}. \tag{3.19}$$

(ii)

$$\begin{aligned}
&(\nabla Y^{n,\frac14}, \nabla\delta Y^n) + c(Y^{n,\frac14}, \delta Y^n) \\
&= \frac{1}{2\Delta t}\{\|\nabla Y^{n+\frac12}\|^2 - \|\nabla Y^{n-\frac12}\|^2\} + \frac{c}{2\Delta t}\{\|Y^{n+\frac12}\|^2 - \|Y^{n-\frac12}\|^2\}.
\end{aligned} \tag{3.20}$$

(iii)

$$\begin{aligned}
&< \overline{Y^{n,\frac14}_{\vec{\nu}_{\Gamma,H}}}, [\delta Y^n] >_\Gamma + < \overline{\delta Y^n_{\vec{\nu}_{\Gamma,H}}}, [Y^{n,\frac14}] >_\Gamma + KH^{-1} < [Y^{n,\frac14}], [\delta Y^n] >_\Gamma \\
&= \frac{1}{2\Delta t}\big\{2 < \overline{Y^{n+\frac12}_{\vec{\nu}_{\Gamma,H}}}, [Y^{n+\frac12}] >_\Gamma + KH^{-1} < [Y^{n+\frac12}], [Y^{n+\frac12}] >_\Gamma \\
&\quad - 2 < \overline{Y^{n-\frac12}_{\vec{\nu}_{\Gamma,H}}}, [Y^{n-\frac12}] >_\Gamma - KH^{-1} < [Y^{n-\frac12}], [Y^{n-\frac12}] >_\Gamma \big\}.
\end{aligned} \tag{3.21}$$

(iv)

$$\begin{aligned}
&< \overline{Y^{n,\frac14}_{\vec{\nu}_{\Gamma,H}}} - \overline{Y^n_{\vec{\nu}_{\Gamma,H}}}, [\delta Y^n] >_\Gamma + < \overline{\delta Y^n_{\vec{\nu}_{\Gamma,H}}}, [Y^{n,\frac14} - Y^n] >_\Gamma + KH^{-1} < [Y^{n,\frac14} - Y^n], [\delta Y^n] >_\Gamma \\
&= \frac{\Delta t}{8}\big\{2 < \overline{\partial_t Y^n_{\vec{\nu}_{\Gamma,H}}}, [\partial_t Y^n] >_\Gamma + KH^{-1} < [\partial_t Y^n], [\partial_t Y^n] >_\Gamma \\
&\quad - 2 < \overline{\partial_t Y^{n-1}_{\vec{\nu}_{\Gamma,H}}}, [\partial_t Y^{n-1}] >_\Gamma - KH^{-1} < [\partial_t Y^{n-1}], [\partial_t Y^{n-1}] >_\Gamma \big\}.
\end{aligned} \tag{3.22}$$

Combining the above equalities (3.19)-(3.22) together, we can see that

$$\|Y^n\|^2_{\mathcal{E}_1} + 2\Delta t < \delta Y^n, \delta Y^n >_{\Gamma_N} = \|Y^{n-1}\|^2_{\mathcal{E}_1}. \tag{3.23}$$

Then, we have estimate (3.16).

Now, we turn to prove estimate (3.17). In equation (3.7), we take $w = -\delta P^n$ to have

$$\begin{aligned}
&(\partial_t^2 P^n, -\delta P^n) + (\nabla P^{n,\frac14}, -\nabla(\delta P^n)) + c(P^{n,\frac14}, -\delta P^n) + < \overline{P^{n,\frac14}_{\vec{\nu}_{\Gamma,H}}}, [-\delta P^n] >_\Gamma \\
&\quad + < \overline{-\delta P^n_{\vec{\nu}_{\Gamma,H}}}, [P^{n,\frac14}] >_\Gamma + KH^{-1} < [P^{n,\frac14}], [-\delta P^n] >_\Gamma + \big\{ < \overline{P^{n,\frac14}_{\vec{\nu}_{\Gamma,H}}} - \overline{P^n_{\vec{\nu}_{\Gamma,H}}}, \\
&\quad [\delta P^n] >_\Gamma + < \overline{\delta P^n_{\vec{\nu}_{\Gamma,H}}}, [P^{n,\frac14} - P^n] >_\Gamma + KH^{-1} < [P^{n,\frac14} - P^n], [\delta P^n] >_\Gamma \big\} \\
&= < \delta P^n, -\delta P^n >_{\Gamma_N}.
\end{aligned} \tag{3.24}$$

By the notations (3.4) and direct calculations to the terms on the left-hand side of equation (3.24), we have the following results.

(1)

$$(\partial_t^2 P^n, -\delta P^n) \quad = \quad -\frac{1}{2\Delta t}\big\{\|\partial_t P^n\|^2 - \|\partial_t P^{n-1}\|^2\big\}. \tag{3.25}$$

(2)

$$\begin{aligned} &(\nabla P^{n,\frac{1}{4}}, -\nabla \delta P^n) + c(P^{n,\frac{1}{4}}, -\delta P^n) \\ &= -\frac{1}{2\Delta t}\big\{\|\nabla P^{n+\frac{1}{2}}\|^2 - \|\nabla P^{n-\frac{1}{2}}\|^2\big\} - \frac{c}{2\Delta t}\big\{\|P^{n+\frac{1}{2}}\|^2 - \|P^{n-\frac{1}{2}}\|^2\big\}. \end{aligned} \tag{3.26}$$

(3)

$$\begin{aligned} &< \overline{P^{n,\frac{1}{4}}_{\vec{\nu}_{\Gamma,H}}}, [-\delta P^n] >_\Gamma + < \overline{-\delta P^n_{\vec{\nu}_{\Gamma,H}}}, [P^{n,\frac{1}{4}}] >_\Gamma + KH^{-1} < [P^{n,\frac{1}{4}}], [-\delta P^n] >_\Gamma \\ &= -\frac{1}{2\Delta t}\big\{2 < \overline{P^{n+\frac{1}{2}}_{\vec{\nu}_{\Gamma,H}}}, [P^{n+\frac{1}{2}}] >_\Gamma + KH^{-1} < [P^{n+\frac{1}{2}}], [P^{n+\frac{1}{2}}] >_\Gamma \\ &\quad -2 < \overline{P^{n-\frac{1}{2}}_{\vec{\nu}_{\Gamma,H}}}, [P^{n-\frac{1}{2}}] >_\Gamma - KH^{-1} < [P^{n-\frac{1}{2}}], [P^{n-\frac{1}{2}}] >_\Gamma\big\}. \end{aligned} \tag{3.27}$$

(4)

$$\begin{aligned} &< \overline{P^{n,\frac{1}{4}}_{\vec{\nu}_{\Gamma,H}}} - \overline{P^n_{\vec{\nu}_{\Gamma,H}}}, [-\delta P^n] >_\Gamma + < \overline{-\delta P^n_{\vec{\nu}_{\Gamma,H}}}, [P^{n,\frac{1}{4}} - P^n] >_\Gamma \\ &\quad + KH^{-1} < [P^{n,\frac{1}{4}} - P^n], [-\delta P^n] >_\Gamma \\ &= -\frac{\Delta t}{8}\big\{2 < \overline{\partial_t P^n_{\vec{\nu}_{\Gamma,H}}}, [\partial_t P^n] >_\Gamma + KH^{-1} < [\partial_t P^n], [\partial_t P^n] >_\Gamma \\ &\qquad -2 < \overline{\partial_t P^{n-1}_{\vec{\nu}_{\Gamma,H}}}, [\partial_t P^{n-1}] >_\Gamma - KH^{-1} < [\partial_t P^{n-1}], [\partial_t P^{n-1}] >_\Gamma\big\}. \end{aligned} \tag{3.28}$$

By the above equalities (3.25)-(3.28), it follows

$$\|P^n\|^2_{\mathcal{E}_1} - 2\Delta t < \delta P^n, \delta P^n >_{\Gamma_N} = \|P^{n-1}\|^2_{\mathcal{E}_1}. \tag{3.29}$$

From this equation, it follows the estimate (3.17). □

Remark 3.2. The results (3.16)-(3.17) show that the full discrete schemes (3.6)-(3.7) keep the conservation of "energy" under the condition $f = 0$, $U = 0$. This means that the schemes (3.6)-(3.7) are stable.

4 Error Estimates

4.1 Auxiliary lemmas

First, the following approximation properties exist.

Lemma 4.1. *[20] For smooth enough function v, there hold estimates*

$$\|\overline{v}_H - v\|_{L^2(\Gamma)} \le \sqrt{2H}\|\nabla v\|_{L^2(\Omega)}, \tag{4.1}$$

$$\|\overline{v}_H - v\|_{L^\infty(\Gamma)} \le CH^2\|v\|_{W^{2,\infty}(\Omega)}, \tag{4.2}$$

and

$$v(x) - \overline{v}_H(x) = -\frac{1}{6}H^2 v_{\vec{\nu}^2_\Gamma}(x) - \frac{1}{120}H^4 v_{\vec{\nu}^4_\Gamma}(x) + o(H^6), \;\; \forall\, x \text{ on } \Gamma, \tag{4.3}$$

where $v_{\vec{\nu}^2_\Gamma}(x)$ and $v_{\vec{\nu}^4_\Gamma}(x)$ are the second and fourth order normal derivatives of v on Γ, respectively.

Define an average operator $\Pi_h : M \to M^h$ on element τ_U satisfying

$$\Pi_h u|_{\tau_U} = \frac{1}{|\tau_U|}\int_{\tau_U} u, \ \forall\, u \in M, \tau_U \in T_U^h, \tag{4.4}$$

where $|\tau_U|$ is the measure of element τ_U.

Lemma 4.2. *[31] For the average operator Π_h, there exists a positive constant C independent of h_U such that*

$$\|\psi - \Pi_h\psi\|_{L^2(\tau_U)} \le Ch_U\|\psi\|_{H^1(\tau_U)}, \ \forall\, \psi \in H^1(\tau_U). \tag{4.5}$$

Similar to the method of reference [32], we introduce an auxiliary problem: Find intermediate variables $\{Y^n(u), P^n(u)\}_{n=1}^N \in V^h \times V^h$, $n = 1, 2, \cdots, N$ satisfying $\forall\, v, w \in V_0^h$

$$\begin{cases} (\partial_t^2 Y^n(u), v) + (\nabla Y^{n,\frac{1}{4}}(u), \nabla v) + c(Y^{n,\frac{1}{4}}(u), v) + < \overline{Y^n_{\vec{\nu}_{\Gamma,H}}}(u), [v] >_\Gamma \\ \quad + < \overline{v_{\vec{\nu}_{\Gamma,H}}}, [Y^n(u)] >_\Gamma + KH^{-1} < [Y^n(u)], [v] >_\Gamma \\ = (f^{n,\frac{1}{4}}, v) + < u^{n,\frac{1}{4}}, v >_{\Gamma_N} - < \delta Y^n(u), v >_{\Gamma_N}, & \text{in } \Omega, \\ Y^0(u) = y_1, & \text{in } \Omega, \\ \partial_t Y^0(u) = y_2, & \text{in } \Omega, \end{cases} \tag{4.6}$$

and

$$\begin{cases} (\partial_t^2 P^n(u), w) + (\nabla P^{n,\frac{1}{4}}(u), \nabla w) + c(P^{n,\frac{1}{4}}(u), w) + < \overline{P^n_{\vec{\nu}_{\Gamma,H}}}(u), [w] >_\Gamma \\ \quad + < \overline{w_{\vec{\nu}_{\Gamma,H}}}, [P^n(u)] >_\Gamma + KH^{-1} < [P^n(u)], [w] >_\Gamma \\ =< \delta P^n(u), w >_{\Gamma_N}, & \text{in } \Omega, \\ P^N(u) = P^N + (Y^N(u) - Y^N), & \text{in } \Omega, \\ \partial_t P^{N-1}(u) = \partial_t P^{N-1} + (\partial_t Y^{N-1}(u) - \partial_t Y^{N-1}), & \text{in } \Omega. \end{cases} \tag{4.7}$$

Let

$$\begin{cases} \theta^n = Y^n - Y^n(u), \quad \eta^n = y^n - Y^n(u), \quad n = 0, 1, 2, \cdots, N, \\ \zeta^n = P^n - P^n(u), \quad \xi^n = p^n - P^n(u), \quad n = N, N-1, \cdots, 0. \end{cases} \tag{4.8}$$

It is clear to see that $\theta^0 = \theta^1 = 0$, $\zeta^N = \theta^N$, $\partial_t\zeta^{N-1} = \partial_t\theta^{N-1}$.

Lemma 4.3. *Let $\{Y^n, P^n\}$ and $\{Y^n(u), P^n(u)\}$ be the solutions of the full discrete schemes* (3.6)-(3.8) *and the auxiliary problem* (4.6)-(4.7)*, respectively. Then, there exists a positive constant C independent of h_U such that*

$$\max_{0\le n\le N}\|\partial_t\theta^n\|^2 + \max_{0\le n\le N}\|\theta^n\|^2 \le C\|u - U\|^2_{l^2(0,T;L^2(\Gamma_N))}, \tag{4.9}$$

$$\max_{0\le n\le N}\|\partial_t\zeta^n\|^2 + \max_{0\le n\le N}\|\zeta^n\|^2 \le C\|u - U\|^2_{l^2(0,T;L^2(\Gamma_N))}, \tag{4.10}$$

where

$$\|u - U\|^2_{l^2(0,T;L^2(\Gamma_N))} \overset{Def.}{=} \sum_{n=1}^{N-1}\Delta t\|u^{n,\frac{1}{4}} - U^{n,\frac{1}{4}}\|^2_{L^2(\Gamma_N)}.$$

Proof. Subtracting equation (4.6) from equation (3.6), we obtain $\forall\, v \in V_0^h$

$$\begin{aligned} &(\partial_t^2\theta^n, v) + (\nabla\theta^{n,\frac{1}{4}}, \nabla v) + c(\theta^{n,\frac{1}{4}}, v) + < \overline{\theta^n_{\vec{\nu}_{\Gamma,H}}}, [v] >_\Gamma + < \overline{v_{\vec{\nu}_{\Gamma,H}}}, [\theta^n] >_\Gamma \\ &+ KH^{-1} < [\theta^n], [v] >_\Gamma =< U^{n,\frac{1}{4}} - u^{n,\frac{1}{4}}, v >_{\Gamma_N} - < \delta\theta^n, v >_{\Gamma_N}. \end{aligned} \tag{4.11}$$

Take $v = \delta\theta^n$ in equation (4.11) to have

$$(\partial_t^2\theta^n, \delta\theta^n) + (\nabla\theta^{n,\frac{1}{4}}, \nabla\delta\theta^n) + c(\theta^{n,\frac{1}{4}}, \delta\theta^n) + < \overline{\theta^n_{\vec{\nu}_{\Gamma,H}}}, [\delta\theta^n] >_\Gamma + < \overline{\delta\theta^n_{\vec{\nu}_{\Gamma,H}}}, [\theta^n] >_\Gamma \\ +KH^{-1} < [\theta^n], [\delta\theta^n] >_\Gamma = < U^{n,\frac{1}{4}} - u^{n,\frac{1}{4}}, \delta\theta^n >_\Gamma - < \delta\theta^n, \delta\theta^n >_{\Gamma_N}. \tag{4.12}$$

By the notations (3.4), we know

$$c(\theta^{n,\frac{1}{4}}, \delta\theta^n) \quad = \quad \frac{c}{2\Delta t}\{\|\theta^{n+\frac{1}{2}}\|^2 - \|\theta^{n-\frac{1}{2}}\|^2\}. \tag{4.13}$$

From the proof of Lemma 3.3, equations (3.12) and (4.13), it follows

$$\text{the left-hand side of equation (4.12)} = \frac{1}{2\Delta t}\{\|\theta^n\|^2_{\mathcal{E}_1} - \|\theta^{n-1}\|^2_{\mathcal{E}_1}\}. \tag{4.14}$$

Analyzing the right-hand side of equation (4.12), we get

$$| < U^{n,\frac{1}{4}} - u^{n,\frac{1}{4}}, \delta\theta^n >_{\Gamma_N} | \leq \varepsilon\{\|\partial_t\theta^n\|^2 + \|\partial_t\theta^{n-1}\|^2\} + C\|U^{n,\frac{1}{4}} - u^{n,\frac{1}{4}}\|^2_{L^2(\Gamma_N)}, \tag{4.15}$$

and

$$| < \delta\theta^n, \delta\theta^n >_{\Gamma_N} | \quad \leq \quad C\{\|\partial_t\theta^n\|^2 + \|\partial_t\theta^{n-1}\|^2\}, \tag{4.16}$$

where $0 < \varepsilon < \frac{1}{4}$ is chosen.

Combining estimates (4.14)-(4.16) together, multiplying both sides by $2\Delta t$, summing time up to n, and using the initial condition $\theta^0 = \theta^1 = 0$, we find

$$\|\theta^n\|^2_{\mathcal{E}_1} \ \leq C\Delta t\sum_{l=1}^{n}\{\|\partial\theta^l\|^2 + \|\partial\theta^{l-1}\|^2\} + C\Delta t\sum_{l=1}^{n}\{\|U^{l,\frac{1}{4}} - u^{l,\frac{1}{4}}\|^2\}. \tag{4.17}$$

Then, from Lemmas 3.1, 3.2 and the discrete Gronwall lemma, it follows

$$\frac{1}{2}\|\partial_t\theta^n\|^2 + C_0|||\theta^{n+\frac{1}{2}}|||^2 + c\|\theta^{n+\frac{1}{2}}\|^2 \ \leq C\Delta t\sum_{l=1}^{n}\|U^{l,\frac{1}{4}} - u^{l,\frac{1}{4}}\|^2_{L^2(\Gamma_N)}. \tag{4.18}$$

Furthermore, since there exist $\theta^{n+1} = \theta^{n+\frac{1}{2}} + \frac{\Delta t}{2}\partial_t\theta^n$ and $\theta^{\frac{1}{2}} = 0$, we achieve

$$\|\theta^{n+1}\| \ \leq C\Delta t\sum_{l=1}^{n}\|\partial_t\theta^l\|. \tag{4.19}$$

Then, estimate (4.9) can be derived by equations (4.17)-(4.18).

Next, we turn to prove estimate (4.10) similarly. Subtracting equation (4.7) from equation (3.7), we see $\forall\, w \in V_0^h$

$$(\partial_t^2\zeta^n, w) + (\nabla\zeta^{n,\frac{1}{4}}, \nabla w) + c(\zeta^{n,\frac{1}{4}}, w) + < \overline{\zeta^n_{\vec{\nu}_{\Gamma,H}}}, [w] >_\Gamma \\ + < \overline{w_{\vec{\nu}_{\Gamma,H}}}, [\zeta^n] >_\Gamma + KH^{-1} < [\zeta^n], [w] >_\Gamma = < \delta\zeta^n, w >_{\Gamma_N}. \tag{4.20}$$

In equation (4.20), taking $w = -\delta\zeta^n$ and similarly to the derivation of equation (4.9), we know

$$\text{the left-hand side of equation (4.20)} \ = \frac{1}{2\Delta t}\{\|\zeta^{n-1}\|^2_{\mathcal{E}_1} - \|\zeta^n\|^2_{\mathcal{E}_1}\}, \tag{4.21}$$

and

$$| < \delta\zeta^n, -\delta\zeta^n >_{\Gamma_N} | \leq C\{\|\partial_t\zeta^n\|^2 + \|\partial_t\zeta^{n-1}\|^2\}. \tag{4.22}$$

Thus, we get

$$\frac{1}{2\Delta t}\{\|\zeta^{n-1}\|^2_{\mathcal{E}_1} - \|\zeta^n\|^2_{\mathcal{E}_1}\} \leq C\{\|\partial_t\zeta^{n-1}\|^2 + \|\partial_t\zeta^n\|^2\}. \tag{4.23}$$

For equation (4.23), multiplying both sides by $2\Delta t$, summing time up to n, we obtain

$$\begin{aligned} \|\zeta^{n-1}\|^2_{\mathcal{E}_1} &\leq \|\zeta^N\|^2_{\mathcal{E}_1} + C\Delta t\sum_{l=n}^{N}\{\|\partial_t\zeta^l\|^2 + \|\partial_t\zeta^{l-1}\|^2\} \\ &\leq \|\theta^N\|^2 + C\Delta t\sum_{l=n}^{N}\{\|\partial_t\zeta^l\|^2 + \|\partial_t\zeta^{l-1}\|^2\}. \end{aligned} \tag{4.24}$$

Then, from Lemmas 3.1, 3.2, the discrete Gronwall lemma and equation (4.9), it follows

$$\frac{1}{2}\|\partial_t\zeta^{n-1}\|^2 + C_0|||\zeta^{n+\frac{1}{2}}|||^2 + c\|\zeta^{n+\frac{1}{2}}\|^2 \leq C\|u - U\|^2_{l^2(0,T;L^2(\Gamma_N))}. \tag{4.25}$$

By $\zeta^{n-1} = \zeta^{n-\frac{1}{2}} - \frac{\Delta t}{2}\partial_t\zeta^{n-1}$, it is easy to bound

$$\|\zeta^{n-1}\| \leq \|\zeta^N\| + \Delta t\sum_{l=n}^{N}\|\partial_t\zeta^l\| \leq \|\theta^N\| + \Delta t\sum_{l=n}^{N}\|\partial_t\zeta^l\|. \tag{4.26}$$

Then, estimate (4.10) follows from equations (4.25)-(4.26). □

4.2 The error estimate of $u - U$

We turn to derive the error estimate of $u - U$ bounded by $p - P(u)$.

Lemma 4.4. *Let $\{y, p, u\}$ and $\{Y, P, U\}$ be the solutions of the optimality system* (2.2)-(2.4) *and the full discrete schemes* (3.6)-(3.8)*, respectively. Assume $u \in L^2(0, T; H^1(\Gamma_N))$, $p \in W$, there exists a positive constant C independent of h_U such that*

$$\|u - U\|^2_{l^2(0,T;L^2(\Gamma_N))} \leq Ch_U^2 + C\|p - P(u)\|^2_{l^2(0,T;L^2(\Omega))}. \tag{4.27}$$

Proof. By direct calculation, we have

$$
\begin{aligned}
& \alpha\|u-U\|^2_{l^2(0,T;L^2(\Gamma_N))} = \alpha\Delta t\sum_{n=1}^{N-1}\|u^{n,\frac14}-U^{n,\frac14}\|^2_{L^2(\Gamma_N)} \\
= \quad & \sum_{n=1}^{N-1}\Delta t < \alpha u^{n,\frac14}+p^{n,\frac14}, u^{n,\frac14}-U^{n,\frac14} >_{\Gamma_N} + \sum_{n=1}^{N-1}\Delta t < \alpha U^{n,\frac14}+P^{n,\frac14}(u), \\
& U^{n,\frac14}-u^{n,\frac14} >_{\Gamma_N} + \Delta t\sum_{n=1}^{N-1} < P^{n,\frac14}(u)-p^{n,\frac14}, u^{n,\frac14}-U^{n,\frac14} >_{\Gamma_N} \\
\leq \quad & \Delta t\sum_{n=1}^{N-1} < \alpha U^{n,\frac14}+P^{n,\frac14}(u), U^{n,\frac14}-u^{n,\frac14} >_{\Gamma_N} \\
& +\Delta t\sum_{n=1}^{N-1} < P^{n,\frac14}(u)-p^{n,\frac14}, u^{n,\frac14}-U^{n,\frac14} >_{\Gamma_N} \\
\leq \quad & \Delta t\sum_{n=1}^{N-1} < \alpha U^{n,\frac14}, \Pi_h u^{n,\frac14}-u^{n,\frac14} >_{\Gamma_N} + \Delta t\sum_{n=1}^{N-1} < p^{n,\frac14}, \Pi_h u^{n,\frac14}-u^{n,\frac14} >_{\Gamma_N} \\
& +\Delta t\sum_{n=1}^{N-1} < p^{n,\frac14}-P^{n,\frac14}(u), u^{n,\frac14}-\Pi_h u^{n,\frac14} >_{\Gamma_N} + \Delta t\sum_{n=1}^{N-1} < P^{n,\frac14}(u)- \\
& P^{n,\frac14}, u^{n,\frac14}-\Pi_h u^{n,\frac14} >_{\Gamma_N} + \Delta t\sum_{n=1}^{N-1} < P^{n,\frac14}(u)-P^{n,\frac14}, U^{n,\frac14}-u^{n,\frac14} >_{\Gamma_N} \\
& +\Delta t\sum_{n=1}^{N-1} < P^{n,\frac14}(u)-p^{n,\frac14}, u^{n,\frac14}-U^{n,\frac14} >_{\Gamma_N} \\
\overset{Def.}{=} \quad & \sum_{i=1}^{6} I_i.
\end{aligned}
\tag{4.28}
$$

Now, we analyze the terms I_1 to I_6 one by one. By the definition of operator Π_h in equation (4.4), there holds

$$
I_1 = \Delta t\sum_{n=1}^{N-1} < \alpha U^{n,\frac14}, \Pi_h u^{n,\frac14}-u^{n,\frac14} >_{\Gamma_N} = 0. \tag{4.29}
$$

From Lemma 4.2, we know

$$
I_2 \quad \leq \quad Ch_U^2\Delta t\sum_{n=1}^{N}(\|p^{n,\frac14}\|^2_{H^1(\Omega)}+\|u^{n,\frac14}\|^2_{H^1(\Gamma_N)}) \leq Ch_U^2, \tag{4.30}
$$

and

$$
I_3 \quad \leq \quad C\|p-P(u)\|^2_{l^2(0,T;L^2(\Omega))}+Ch_U^2. \tag{4.31}
$$

By Lemmas 4.2 and 4.3, we see

$$
I_4 \quad \leq \quad \varepsilon\|U-u\|^2_{l^2(0,T;L^2(\Gamma_N))}+Ch_U^2, \tag{4.32}
$$

where ε is an arbitrary positive constant.

It is now to bound the term I_5. In equation (4.11), we take $v=\zeta^{n,\frac14}$ to obtain

$$
\begin{aligned}
& (\partial_t^2\theta^n, \zeta^{n,\frac14}) + (\nabla\theta^{n,\frac14}, \nabla\zeta^{n,\frac14}) + c(\theta^{n,\frac14}, \zeta^{n,\frac14}) + < \overline{\theta^n_{\vec{\nu}_{\Gamma,H}}}, [\zeta^{n,\frac14}] >_\Gamma + < \overline{\zeta^{n,\frac14}_{\vec{\nu}_{\Gamma,H}}}, [\theta^n] >_\Gamma \\
& +KH^{-1} < [\theta^n], [\zeta^{n,\frac14}] >_\Gamma = < U^{n,\frac14}-u^{n,\frac14}, \zeta^{n,\frac14} >_{\Gamma_N} - < \delta\theta^n, \zeta^{n,\frac14} >_{\Gamma_N}.
\end{aligned}
\tag{4.33}
$$

By $\theta^0 = \theta^1 = 0, \zeta^N = \theta^N, \partial_t \zeta^{N-1} = \partial_t \theta^N$, sum time up to n and utilize the discrete Green formula to have
(1)

$$\Delta t \sum_{n=1}^{N-1} (\partial_t^2 \theta^n, \zeta^{n,\frac{1}{4}}) = \Delta t \sum_{n=1}^{N-1} (\partial_t^2 \zeta^n, \theta^{n,\frac{1}{4}}), \tag{4.34}$$

(2)

$$\begin{aligned} & \sum_{n=1}^{N-1} \Delta t \{ < \overline{\theta^n_{\vec{\nu}_{\Gamma,H}}}, [\zeta^{n,\frac{1}{4}}] >_\Gamma + < \overline{\zeta^{n,\frac{1}{4}}_{\vec{\nu}_{\Gamma,H}}}, [\theta^n] >_\Gamma + KH^{-1} < [\theta^n], [\zeta^{n,\frac{1}{4}}] >_\Gamma \} \\ = & \sum_{n=1}^{N-1} \Delta t \{ < \overline{\theta^{n,\frac{1}{4}}_{\vec{\nu}_{\Gamma,H}}}, [\zeta^n] >_\Gamma + < \overline{\zeta^n_{\vec{\nu}_{\Gamma,H}}}, [\theta^{n,\frac{1}{4}}] >_\Gamma + KH^{-1} < [\theta^{n,\frac{1}{4}}], [\zeta^n] >_\Gamma \}. \end{aligned} \tag{4.35}$$

Hence, we find

$$\begin{aligned} I_5 &= -\sum_{n=1}^{N-1} \Delta t < \zeta^{n,\frac{1}{4}}, U^{n,\frac{1}{4}} - u^{n,\frac{1}{4}} >_{\Gamma_N} \\ &= -\sum_{n=1}^{N-1} \Delta t \{ (\partial_t^2 \theta^n, \zeta^{n,\frac{1}{4}}) + (\nabla \theta^{n,\frac{1}{4}}, \nabla \zeta^{n,\frac{1}{4}}) + c(\theta^{n,\frac{1}{4}}, \zeta^{n,\frac{1}{4}}) + < \overline{\theta^n_{\vec{\nu}_{\Gamma,H}}}, [\zeta^{n,\frac{1}{4}}] >_\Gamma \\ &\quad + < \overline{\zeta^{n,\frac{1}{4}}_{\vec{\nu}_{\Gamma,H}}}, [\theta^n] >_\Gamma + KH^{-1} < [\theta^n], [\zeta^{n,\frac{1}{4}}] >_\Gamma + < \delta\theta^n, \zeta^{n,\frac{1}{4}} >_{\Gamma_N} \} \\ &= -\sum_{n=1}^{N-1} \big\{ (\partial_t^2 \zeta^n, \theta^{n,\frac{1}{4}}) + (\nabla \zeta^{n,\frac{1}{4}}, \nabla \theta^{n,\frac{1}{4}}) + c(\zeta^{n,\frac{1}{4}}, \theta^{n,\frac{1}{4}}) + < \overline{\zeta^n_{\vec{\nu}_{\Gamma,H}}}, [\theta^{n,\frac{1}{4}}] >_\Gamma \\ &\quad + < \overline{\theta^{n,\frac{1}{4}}_{\vec{\nu}_{\Gamma,H}}}, [\zeta^n] >_\Gamma + KH^{-1} < [\zeta^n], [\theta^{n,\frac{1}{4}}] >_\Gamma + < \delta\theta^n, \zeta^{n,\frac{1}{4}} >_{\Gamma_N} \big\} \\ &= -\sum_{n=1}^{N-1} \Delta t \{ < \delta\zeta^n, \theta^{n,\frac{1}{4}} >_{\Gamma_N} + < \delta\theta^n, \zeta^{n,\frac{1}{4}} >_{\Gamma_N} \} \\ &= - < \zeta^{N-\frac{1}{2}}, \theta^{N-\frac{1}{2}} >_{\Gamma_N} \\ &\leq 0. \end{aligned} \tag{4.36}$$

It is easy to bound the last term as

$$I_6 \leq C\|P(u) - p\|^2_{l^2(0,T;L^2(\Omega))} + \varepsilon \|u - U\|^2_{l^2(0,T;L^2(\Gamma_N))}. \tag{4.37}$$

By estimates (4.28)-(4.37), taking $\varepsilon \leq \dfrac{\alpha}{4}$, we derive

$$\|u - U\|^2_{l^2(0,T;L^2(\Gamma_N))} \leq Ch_U^2 + C\|p - P(u)\|^2_{l^2(0,T;L^2(\Omega))}. \tag{4.38}$$

Then, the proof of Lemma 4.4 is completed. □

4.3 The error estimates of $y - Y(u)$ and $p - P(u)$

We adopt the standard elliptic projection $\tilde{w} \in V^h$ of $w \in V$

$$(\nabla(w - \tilde{w}), \nabla v) + c(w - \tilde{w}, v) = 0, \quad \forall\ v \in V^h. \tag{4.39}$$

By [33][34][35][36], we know that

Lemma 4.5. *For elliptic projection* (4.39)*, there exists the following* L^∞*-norm error estimate*

$$\begin{aligned}&\|w-\tilde{w}\|_{L^\infty(\Omega)}+\|(w-\tilde{w})_t\|_{L^\infty(\Omega)}+\|(w-\tilde{w})_{tt}\|_{L^\infty(\Omega)}\\ \leq\quad & Ch^2|\ln h|\{\|w\|_{W^{2,\infty}(\Omega)}+\|w_t\|_{W^{2,\infty}(\Omega)}+\|w_{tt}\|_{W^{2,\infty}(\Omega)}\}.\end{aligned}\tag{4.40}$$

As we have shown, the full discrete schemes (3.6) and (3.8) include two special terms on the inter-domain boundary Γ by integral mean method to present explicit flux calculation. The standard elliptic projection (4.39) is insufficient for deriving optimal error estimates. To get optimal error estimates, a new elliptic projection which includes the terms of inter-domain boundary was introduced in [20]. This new elliptic projection $w_I \in V^h$ of the solution $w \in V$ is defined as follows: $\forall\, v \in V_0^h$

$$\begin{aligned}&(\nabla(w-w_I),\nabla v)+c(w-w_I,v)+<\overline{(w-w_I)_{\vec{\nu}_\Gamma,H}},[v]>_\Gamma\\ &+<\overline{v_{\vec{\nu}_\Gamma,H}},[w-w_I]>_\Gamma+KH^{-1}<[w-w_I],[v]>_\Gamma=0.\end{aligned}\tag{4.41}$$

Lemma 4.6. *[20] For* $w-w_I$*, there exists error estimate*

$$\begin{cases}\|w-w_I\|\ \leq\ C\{h^2+H^{1/2}\|w-\tilde{w}\|_{L^\infty}\},\\ \|(w-w_I)_t\|\ \leq\ C\{h^2+H^{1/2}\|(w-\tilde{w})_t\|_{L^\infty}\},\\ \|(w-w_I)_{tt}\|\ \leq\ C\{h^2+H^{1/2}\|(w-\tilde{w})_{tt}\|_{L^\infty}\},\\ \|\nabla(w-w_I)_{tt}\|\ \leq\ C\{h+H^{1/2}\|\nabla(w-\tilde{w})_{tt}\|_{L^\infty}\},\end{cases}\tag{4.42}$$

where $\tilde{w}$ *is the elliptic projection defined by equation* (4.39)*.*

Lemma 4.7. *Let* $\{y,p\},\{Y(u),P(u)\}$ *be the solutions of the optimality system* (2.2)-(2.4) *and auxiliary problem* (4.6)-(4.7)*, respectively. Supposing that* $y,p\in L^2(0,T;H^1(\Omega))\cap H^2(0,T;L^2(\Omega))$ *and* $H=O(h^{\frac{4}{5}})$*, there exists a positive constant* C *independent of* h*,* H *and* Δt *such that*

$$\max_{1\leq n\leq N}\|\partial_t\eta^n\|+\max_{1\leq n\leq N}\|\eta^n\|\ \leq\ C\{(\Delta t)^2+h^2+H^{5/2}\},\tag{4.43}$$

$$\max_{1\leq n\leq N}\|\partial_t\xi^n\|+\max_{1\leq n\leq N}\|\xi^n\|\ \leq\ C\{(\Delta t)^2+h^2+H^{5/2}\},\tag{4.44}$$

provided that $\Delta t\leq C_1H$*, where constant* C_1 *is defined by Lemma 3.2.*

Proof. **Part 1.** By equation (2.2), we have the weak formulation $\forall\, v\in V_0^h$

$$\begin{aligned}&(\partial_t^2y^n,v)+(\nabla y^{n,\frac14},\nabla v)+c(y^{n,\frac14},v)+<y_{\vec{\nu}_\Gamma}^{n,\frac14},[v]>_\Gamma\\ =\quad&(f^{n,\frac14},v)+<u^{n,\frac14}-\partial_ty^{n,\frac14},v>_{\Gamma_N}+(\partial_t^2y^n-\frac{\partial^2y^{n,\frac14}}{\partial t^2},v),\end{aligned}\tag{4.45}$$

where

$$\|\partial_t^2y^n-\frac{\partial^2y^{n,\frac14}}{\partial t^2}\|^2\ \leq\ (\Delta t)^3\int_{t^{n-1}}^{t^{n+1}}\|\frac{\partial^4y}{\partial t^4}\|dt\ \leq\ C(\Delta t)^3\tag{4.46}$$

follows [37].

Let $y_I\in V^h$ is the elliptic projection of y by equation (4.41). Define $\beta=y-y_I$, $\rho=Y(u)-y_I$. Combining equations (4.41) and (4.45) together, we know

$$\begin{aligned}&(\partial_t^2y_I^n,v)+(\nabla y_I^{n,\frac14},\nabla v)+c(y_I^{n,\frac14},v)+<\overline{y_{I,\vec{\nu}_\Gamma,H}^n},[v]>_\Gamma\\ &+<\overline{v_{\vec{\nu}_\Gamma,H}},[y_I^n]>_\Gamma+KH^{-1}<[y_I^n],[v]>_\Gamma\\ =\quad&(f^{n,\frac14},v)+<u^{n,\frac14},v>_{\Gamma_N}+<\delta y^n-\partial_ty^{n,\frac14}>_{\Gamma_N}+(\partial_t^2y^n-\frac{\partial^2y^{n,\frac14}}{\partial t^2},v)\\ &+<\overline{y_{\vec{\nu}_\Gamma,H}^n}-y_{\vec{\nu}_\Gamma}^n,[v]>_\Gamma+<y_{\vec{\nu}_\Gamma}^n-y_{\vec{\nu}_\Gamma}^{n,\frac14},[v]>_\Gamma-<\delta\beta^n,v>_{\Gamma_N}\\ &-(\nabla(\beta^{n,\frac14}-\beta^n),\nabla v)-(\partial_t^2\beta^n,v)-c(\beta^{n,\frac14},v)-<\delta y_I^n,v>_{\Gamma_N}.\end{aligned}\tag{4.47}$$

Subtracting equation (4.47) from equations (4.6), we obtain

$$\begin{aligned}
& (\partial_t^2\rho^n, v) + (\nabla\rho^{n,\frac{1}{4}}, \nabla v) + c(\rho^{n,\frac{1}{4}}, v) + < \overline{\rho^n_{\vec{\nu}_{\Gamma,H}}}, [v] >_\Gamma \\
& + < \overline{v_{\vec{\nu}_{\Gamma,H}}}, [\rho^n] >_\Gamma + KH^{-1} < [\rho^n], [v] >_\Gamma \\
= \; & (\nabla(\beta^{n,\frac{1}{4}} - \beta^n), \nabla v) + c(\beta^{n,\frac{1}{4}}, v) - < \overline{y^n_{\vec{\nu}_{\Gamma,H}}} - y^n_{\vec{\nu}_{\Gamma,H}}, [v] >_\Gamma \\
& - < y^n_{\vec{\nu}_{\Gamma,H}} - y^{n,\frac{1}{4}}_{\vec{\nu}_\Gamma}, [v] >_\Gamma + (\partial_t^2\beta^n, v) - (\partial_t^2 y^n - \frac{\partial^2 y^n}{\partial t^2}, v) \\
& + < \delta\beta^n, v >_{\Gamma_N} - < \delta y^n - \partial_t y^{n,\frac{1}{4}} >_{\Gamma_N} - < \delta\rho^n, v >_{\Gamma_N} .
\end{aligned} \tag{4.48}$$

Taking $v = \delta\rho^n$ in equation (4.48), we analyze the terms one by one. Similarly to the derivation of equation (4.14), it follows

$$\text{the left-hand side of equation (4.48)} = \frac{1}{2\Delta t}\{\|\rho^n\|^2_{\mathcal{E}_1} - \|\rho^{n-1}\|^2_{\mathcal{E}_1}\}. \tag{4.49}$$

It is easy to see that the following estimates exist

$$\begin{aligned}
|(\nabla(\beta^{n,\frac{1}{4}} - \beta^n), \nabla\delta\rho^n)| &\leq \varepsilon\{\|\nabla\rho^{n+\frac{1}{2}}\|^2 + \|\nabla\rho^{n-\frac{1}{2}}\|^2\} + C(\Delta t)^2\|\nabla\partial_t^2\beta^n\|^2, \\
|c(\beta^{n,\frac{1}{4}}, \delta\rho^n)| &\leq C\{\|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\} + C\{\|\beta^{n+\frac{1}{2}}\|^2 + \|\beta^{n-\frac{1}{2}}\|^2\}, \\
| < \delta\beta^n, \delta\rho^n >_{\Gamma_N} | &\leq C\{\|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\} + C\{\|\partial_t\beta^n\|^2 + \|\partial_t\beta^{n-1}\|^2\}, \\
| < \delta\rho^n, \delta\rho^n >_{\Gamma_N} | &\leq C\{\|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\}, \\
|(\partial_t^2\beta^n, \delta\rho^n)| &\leq \frac{\varepsilon}{2}\{\|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\} + C\|\partial_t^2\beta^n\|^2, \\
|(\partial_t^2 y^n - \frac{\partial^2 y^{n,\frac{1}{4}}}{\partial t^2}, \delta\rho^n)| &\leq \frac{\varepsilon}{2}\{\|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\} + C(\Delta t)^3,
\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
& | < y^{n,\frac{1}{4}}_{\vec{\nu}_\Gamma} - y^n_{\vec{\nu}_\Gamma}, [\delta\rho^n] >_\Gamma | \\
\leq \; & \frac{1}{2\Delta t}\big\{\frac{\mu}{2}KH^{-1}\{\|[\rho^{n+\frac{1}{2}}]\|^2_{L^2(\Gamma)} + \|[\rho^{n-\frac{1}{2}}]\|^2_{L^2(\Gamma)}\} + CH^5\|y_{tt}(\tilde{t})\|^2_{H^2(\Omega)}\big\},
\end{aligned} \tag{4.51}$$

where, $\tilde{t} \in (t^{n-1}, t^{n+1})$, arbitrary constants $\varepsilon > 0$ and $0 < \mu < 1$.

Noticing that there exists

$$|\frac{\partial y^{n,\frac{1}{4}}}{\partial t} - \delta y^n| = |\frac{1}{4}\int_{-\Delta t}^{\Delta t}(\Delta t - |\tau|)\frac{|\tau|}{\Delta t}\frac{\partial^3 y}{\partial t^3}(t^n + \tau)d\tau| \leq C(\Delta t)^3. \tag{4.52}$$

Then, we know

$$| < \frac{\partial y^{n,\frac{1}{4}}}{\partial t} - \delta y^n, \delta\rho^n >_{\Gamma_N} | \leq C\{\|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\} + C(\Delta t)^3. \tag{4.53}$$

From Lemma 4.1, we also know

$$\begin{aligned}
& | < \overline{y^n_{\vec{\nu}_{\Gamma,H}}} - y^n_{\vec{\nu}_\Gamma}, [\delta\rho^n] >_\Gamma | \\
\leq \; & \frac{1}{2\Delta t}\big\{CH^5\|y^n_{\vec{\nu}_\Gamma}\|^2_{W^{2,\infty}(\Omega)} + \frac{\mu}{2}KH^{-1}\{\|[\rho^{n+\frac{1}{2}}]\|^2_{L^2(\Gamma)} + \|[\rho^{n-\frac{1}{2}}]\|^2_{L^2(\Gamma)}\}\big\}.
\end{aligned} \tag{4.54}$$

Collecting the above analysis together, and noticing $0 < \mu < 1$, we find

$$\begin{aligned} &(1-\mu)\|\rho^n\|_{\mathcal{E}_1}^2 - (1+\mu)\|\rho^{n-1}\|_{\mathcal{E}_1}^2 \\ \leq\ & C\Delta t\{\|\nabla\rho^{n+\frac{1}{2}}\|^2 + \|\nabla\rho^{n-\frac{1}{2}}\|^2 + \|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\} \\ &+C\Delta t\{(\Delta t)^3 + \|\partial_t^2\beta^n\|^2 + (\Delta t)^2\|\nabla\partial_t^2\beta^n\|^2 + \|\beta^{n+\frac{1}{2}}\|^2 + \|\beta^{n-\frac{1}{2}}\|^2 \\ &+\|\partial_t\beta^n\|^2 + \|\partial_t\beta^{n-1}\|^2\} + CH^5. \end{aligned} \tag{4.55}$$

Define $\lambda = \frac{1-\mu}{1+\mu}$. Then, we see $0 < \lambda < 1$. Multiplying both sides of equation (4.55) by $\frac{\lambda^{n-1}}{1+\mu}$, we have

$$\begin{aligned} &\lambda^n\|\rho^n\|_{\mathcal{E}_1}^2 - \lambda^{n-1}\|\rho^{n-1}\|_{\mathcal{E}_1}^2 \\ \leq\ & C\Delta t\{\|\nabla\rho^{n+\frac{1}{2}}\|^2 + \|\nabla\rho^{n-\frac{1}{2}}\|^2 + \|\partial_t\rho^n\|^2 + \|\partial_t\rho^{n-1}\|^2\} \\ &+C\Delta t\{(\Delta t)^3 + \|\partial_t^2\beta^n\|^2 + (\Delta t)^2\|\nabla\partial_t^2\beta^n\|^2 + \|\beta^{n+\frac{1}{2}}\|^2 + \|\beta^{n-\frac{1}{2}}\|^2 \\ &+\|\partial_t\beta^n\|^2 + \|\partial_t\beta^{n-1}\|^2\} + CH^5. \end{aligned} \tag{4.56}$$

Summing time up to n, we obtain

$$\begin{aligned} \lambda^n\|\rho^n\|_{\mathcal{E}_1}^2 \leq\ & 2C\Delta t\frac{\lambda^{n-1}}{1+\mu}\sum_{l=1}^{n}\big\{\|\nabla\rho^{l+\frac{1}{2}}\|^2 + \|\nabla\rho^{l-\frac{1}{2}}\|^2 + \|\partial_t\rho^l\|^2 + \|\rho^{l-1}\|^2\big\} \\ &+C\Delta t\sum_{l=1}^{n}\big\{(\Delta t)^3 + \|\partial_t^2\beta^l\|^2 + (\Delta t)^2\|\nabla\partial_t^2\beta^l\|^2 + \|\beta^{l+\frac{1}{2}}\|^2 \\ &+\|\beta^{l-\frac{1}{2}}\|^2 + \|\partial_t\beta^l\|^2 + \|\partial_t\beta^{l-1}\|^2\big\} + CH^5. \end{aligned} \tag{4.57}$$

Choosing $\mu = \frac{1}{1+N}$, we know $\lambda^{-n} = (1+\frac{2}{N})^n < e^2$. By the discrete Gronwall lemma, Lemmas 3.1 and 3.2, there holds

$$\begin{aligned} &\frac{1}{2}\|\partial_t\rho^n\|^2 + C_0|||\rho^n|||^2 + c\|\rho^{n+\frac{1}{2}}\|^2 \\ \leq\ & C\Delta t\sum_{l=1}^{n}\{(\Delta t)^3 + \|\partial_t^2\beta^l\|^2 + (\Delta t)^2\|\nabla\partial_t^2\beta^l\|^2 + \|\beta^{l+\frac{1}{2}}\|^2 + \|\beta^{l-\frac{1}{2}}\|^2 \\ &+\|\partial_t\beta^l\|^2 + \|\partial_t\beta^{l-1}\|^2\} + CH^5. \end{aligned} \tag{4.58}$$

Using the equality $\partial_t^2\beta^l = (\Delta t)^{-2}\int_{-\Delta t}^{\Delta t}(\Delta t - |\tau|)\frac{\partial^2\beta}{\partial t^2}(t^l+\tau)d\tau$ ([37]), we derive

$$\left\{\begin{aligned} \Delta t\sum_{l=1}^{n}\|\partial_t^2\beta^l\|^2 &\leq C\|\frac{\partial^2\beta}{\partial t^2}\|_{L^2(0,T;L^2(\Omega))}^2, \\ \Delta t\sum_{l=1}^{n}\|\nabla\partial_t^2\beta^l\|^2 &\leq C\|\frac{\partial^2\beta}{\partial t^2}\|_{L^2(0,T;H^1(\Omega))}^2. \end{aligned}\right. \tag{4.59}$$

By the equality $\rho^{n+1} = \rho^{n+\frac{1}{2}} + \frac{\Delta t}{2}\partial_t\rho^n$ and $\rho^{\frac{1}{2}} = 0$, there exists

$$\|\rho^{n+1}\| \leq \Delta t\sum_{l=1}^{n}\|\partial_t\rho^l\|. \tag{4.60}$$

Putting equations (4.58)-(4.60) together and using Lemma 4.6, we can obtain estimate (4.43).

Part 2. This next step is to estimate the error between p and $P(u)$. The weak formulation of equation (2.3) is $\forall w \in V_0^h$

$$(\partial_t^2 p^n, w) + (\nabla p^{n,\frac{1}{4}}, \nabla w) + c(p^{n,\frac{1}{4}}, w) + < p_{\vec{\nu}_\Gamma}^{n,\frac{1}{4}}, [w] >_\Gamma = < d_t p^{n,\frac{1}{4}}, w >_{\Gamma_N} + (\partial_t^2 p^n - \frac{\partial^2 p^{n,\frac{1}{4}}}{\partial t^2}, w), \tag{4.61}$$

where $\|\partial_t^2 p^n - \frac{\partial^2 p^{n,\frac{1}{4}}}{\partial t^2}\|^2 \le C(\Delta t)^3$.

Define the elliptic projection p_I of p by equation (4.41). Let $\varphi = p - p_I, \psi = P(u) - p_I$. There exists

$$\begin{aligned}
&(\partial_t^2 p_I^n, w) + (\nabla p_I^{n,\frac{1}{4}}, \nabla w) + c(p^{n,\frac{1}{4}}, w) \\
&+ < \overline{p_{I,\vec{\nu}_\Gamma,H}^n}, [w] >_\Gamma + < \overline{w_{\vec{\nu}_\Gamma,H}}, [p_I^n] >_\Gamma + KH^{-1} < [p_I^n], [w] >_\Gamma \\
= \; & < d_t p^{n,\frac{1}{4}} - \delta p^n, w >_{\Gamma_N} + (\partial_t^2 p^n - \frac{\partial^2 p^{n,\frac{1}{4}}}{\partial t^2}, w) - < p_{\vec{\nu}}^{n,\frac{1}{4}} - p_{\vec{\nu}}^n, [w] >_\Gamma \\
&+ < \overline{p_{\vec{\nu},H}^n} - p_{\vec{\nu}_\Gamma}^n, [w] >_\Gamma - (\partial_t^2 \varphi^n, w) - (\nabla(\varphi^{n,\frac{1}{4}} - \varphi^n), \nabla w) \\
&- c(\varphi^{n,\frac{1}{4}}, w) + < \delta\varphi^n, w >_{\Gamma_N} + < \delta p_I^n, w >_{\Gamma_N} .
\end{aligned} \tag{4.62}$$

Subtracting equation (4.62) from equation (4.7), we have

$$\begin{aligned}
&(\partial_t^2 \psi^n, w) + (\nabla \psi^{n,\frac{1}{4}}, \nabla w) + c(\psi^{n,\frac{1}{4}}, w) \\
&+ < \overline{\psi_{\vec{\nu}_\Gamma,H}^n}, [w] >_\Gamma + < \overline{w_{\vec{\nu}_\Gamma,H}}, [\psi^n] >_\Gamma + KH^{-1} < [\psi^n], [w] >_\Gamma \\
= \; & < \delta p^n - \frac{dp^{n,\frac{1}{4}}}{dt}, w >_{\Gamma_N} - (\partial_t^2 p^n - \frac{\partial^2 p^{n,\frac{1}{4}}}{\partial t^2}, w) + < p_{\vec{\nu}_\Gamma,H}^{n,\frac{1}{4}} - p_{\vec{\nu}_\Gamma}^n, [w] >_\Gamma \\
&- < \overline{p_{\vec{\nu}_\Gamma,H}^n} - p_{\vec{\nu}_\Gamma}^n, [w] >_\Gamma + (\partial_t^2 \varphi^n, w) + (\nabla(\varphi^{n,\frac{1}{4}} - \varphi^n), \nabla w) \\
&- < \delta\varphi^n, w >_{\Gamma_N} + < \delta\psi^n, w >_\Gamma + c(\varphi^{n,\frac{1}{4}}, w) + < \frac{dp^{n,\frac{1}{4}}}{dt} - d_t p^{n,\frac{1}{4}}, w >_{\Gamma_N} .
\end{aligned} \tag{4.63}$$

Taking $w = -\delta\psi^n$ in equation (4.63), similarly to the derivation of equation (4.14), we have

$$\text{the left-hand side of equation (4.63)} = \frac{1}{2\Delta t}\{\|\psi^{n-1}\|_{\mathcal{E}_1}^2 - \|\psi^n\|_{\mathcal{E}_1}^2\}. \tag{4.64}$$

We analyze the terms on the right-hand side of equation (4.63) one by one to obtain

$$|c(\varphi^{n,\frac{1}{4}}, -\delta\psi^n)| \le C\{\|\partial_t\psi^n\|^2 + \|\partial_t\psi^{n-1}\|^2\} + C\{\|\varphi^{n+\frac{1}{2}}\|^2 + \|\varphi^{n-\frac{1}{2}}\|^2\}, \tag{4.65}$$

$$| < \delta\varphi^n, -\delta\psi^n >_{\Gamma_N} | \le C\{\|\partial_t\varphi^n\|^2 + \|\partial_t\varphi^{n-1}\|^2\} + C\{\|\partial_t\psi^n\|^2 + \|\partial_t\psi^{n-1}\|^2\}, \tag{4.66}$$

$$| < \delta\psi^n, -\delta\psi^n >_{\Gamma_N} | \le C\{\|\partial_t\psi^n\|^2 + \|\partial_t\psi^{n-1}\|^2\}, \tag{4.67}$$

$$| < \frac{dp^{n,\frac{1}{4}}}{dt} - d_t p^{n,\frac{1}{4}}, -\delta\psi^n >_{\Gamma_N} | \le C\{\|\partial_t\psi^n\|^2 + \|\partial_t\psi^{n-1}\|^2\}, \tag{4.68}$$

$$| < \delta p^n - \frac{p^{n,\frac{1}{4}}}{dt}, -\delta\psi^n >_\Gamma | \le C(\Delta t)^3 + C\{\|\partial_t\psi^n\|^2 + \|\partial_t\psi^{n-1}\|^2\} \tag{4.69}$$

from equation (4.53).

Combining the above analysis, and by the same argument to derive equation (4.57), we see

$$\begin{aligned}\lambda^n\|\psi^{n-1}\|^2_{\mathcal{E}_1} \quad \leq \quad & C\|\rho^N\|^2_{\mathcal{E}_1} + C\Delta t\sum_{l=n}^{N}\{\|\partial_t\psi^l\|^2 + \|\partial_t\psi^{l-1}\|^2 + \|\nabla\psi^{l-\frac{1}{2}}\|^2 + \|\nabla\psi^{l+\frac{1}{2}}\|^2\} \\ & + C\Delta t\sum_{l=n}^{N}\{(\Delta t)^3 + \|\partial_t^2\varphi^l\|^2 + (\Delta t)^2\|\nabla\partial_t^2\varphi^l\|^2 + \|\varphi^{l+\frac{1}{2}}\|^2 + \|\varphi^{l-\frac{1}{2}}\|^2 \\ & + \|\partial_t\varphi^l\|^2 + \|\partial_t\varphi^{l-1}\|^2\} + CH^5.\end{aligned} \tag{4.70}$$

Similarly to Part 1, by the discrete Gronwall lemma and Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned} & \frac{1}{2}\|\partial_t\psi^{n-1}\|^2 + C_0|||\psi^{n+\frac{1}{2}}|||^2 + c\|\psi^{n+\frac{1}{2}}\|^2 \\ \leq \quad & C\|\rho^N\|^2_{\mathcal{E}_1} + C\Delta t\sum_{l=n}^{N}\{(\Delta t)^3 + \|\partial_t^2\varphi^l\|^2 + (\Delta t)^2\|\nabla\partial_t^2\varphi^l\|^2 \\ & + \|\varphi^{l+\frac{1}{2}}\|^2 + \|\varphi^{l-\frac{1}{2}}\|^2 + \|\partial_t\varphi^l\|^2 + \|\partial_t\varphi^{l-1}\|^2\} + CH^5.\end{aligned} \tag{4.71}$$

Hence, similarly to prove Lemma 3.2 and equation (4.43), we finally prove estimate (4.44) exist. □

By the results of Lemmas 4.3, 4.4 and 4.7, we obtain the following error theorem.

Theorem 4.8. *Let $\{y, p, u\}$, $\{Y, P, U\}$ be the solutions of the optimality system* (2.2)-(2.4) *and the full discrete schemes* (3.6)-(3.8)*, respectively. Assuming that the conditions in Lemmas 4.3, 4.4 and 4.7 be held, there exists a positive constant C independent of h, h_U and Δt such that*

$$\max_{1\leq n\leq N}\|\partial_t(y-Y)^n\| + \max_{1\leq n\leq N}\|y^n - Y^n\| \leq C\{h_U + h^2 + H^{5/2} + (\Delta t)^2\}, \tag{4.72}$$

$$\max_{1\leq n\leq N}\|\partial_t(p-P)^n\| + \max_{1\leq n\leq N}\|p^n - P^n\| \leq C\{h_U + h^2 + H^{5/2} + (\Delta t)^2\}, \tag{4.73}$$

provided that $\Delta t \leq C_1 H$, where constant C_1 is defined by Lemma 3.2.

Remark 4.1. From Theorem 4.8, we can know that the full discrete schemes (3.6)-(3.8) have convergence orders on Δt, h and H as same as that of [24]. Since the schemes use implicit Galerkin methods in the sub-domains and explicit flux calculations on the inter-domain boundary Γ by an integral mean method. The time step constraint $\Delta t \leq C_1 H$ in Lemma 3.2 is still needed to preserve stability, which is similar to that of reference work [37].

5 Conclusions

We have presented a non-overlapping DDM to solve optimal boundary control problems governed by wave equations with absorbing boundary condition. An integral mean method is utilized to present an explicit flux calculation on the inter-domain boundary in order to communicate the local problems on the interfaces between subdomains, which helps to compute the local problems on each subdomain fully parallel. We establish the full discrete schemes for solving these local problems, and prove the stability of the schemes. In Theorem 4.8, a priori error estimates are derived for the state, co-state and control variables that show the full discrete schemes (3.6)-(3.8) have convergence orders on Δt, h and H as same as that of [24].

Acknowledgement

This work was supported by the Natural Science Foundation of China (Nos. 11301300, 11271231).

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Clason C, Kaltenbacher B, Veljovic S. Boundary optimal optimal control of the Westervelt and the Kuznetsov equation[J]. Journal of Mathematical Analysis. 2009;356:738-751.

[2] Komatitsch D, Tromp J. Introduction to the spectral ellement method for three dimemsional seismic wave ropagation[J]. Geophysical Journal International. 1999;139:806-822.

[3] Banks HT, Keeling SL, Silcox RJ. Optimal control techniques for actice noise suppression[C]. Proceedings of the 27th Conference on Dicision and Control, Austin, Texas. 1988;3:2006-2011.

[4] Kröner A, Kunisch K, Vexler B. Semi-smooth Newton methods for optimal control of the wave equation with control constraints[J]. SIAM Journal on Control and Optimization. 2011;49:830-858.

[5] Gugat M, Keimer A. Optimal distributed control of the wave equation subject to state constraints[J]. ZAMM - Journal of Applied Mathematics and Mechanics. 2009;89(6):420-444.

[6] Bensoussan A, Glowinski R, Lions JL. Methode de decomposition appliquee au controle optimal de systemes distribues[J]. Lecture Notes in Computer Science. 1973;3:141-151.

[7] Leugering G. Dynamic domain decomposition of optimal control problems for networks of strings and Timoshenko beams[J]. SIAM Journal on Control and Optimization. 1999;37(6):1649-1675.

[8] Leugering G. Domain decomposition of optimal control problems for dynamic networks of elastic strings[J]. Computational Optimization and Applications. 2000;16(1):5-27.

[9] Benamou JD. Domain decomposition methods with coupled transmission conditions for the optimal control of systems governed by elliptic partial differential equations[J]. SIAM journal on numerical analysis. 1996;33(6):2401-2416.

[10] Hou LS, Lee J. A Robin-Robin non-overlapping domain decomposition method for an elliptic boundary control problem[J]. International Journal of Numerical Analysis & Modeling. 2011;8(3):443-465.

[11] Benamou JD. Domain decomposition, optimal control of system governed by partial differential equations, and Sysnthesis of feedback laws[J]. Journal of optimization theory and applications. 1999;102(1):15-36.

[12] Roache PJ. Computational fluid dynamics[M]. Hermosa Publishers; 1972.

[13] Elvius T, Sundstrom A. Computationally efficient schemes and boundary conditions for a fine mesh barotropic model based on the shallow water equations[J]. Tellus. 1973;25:132-156.

[14] Boore DM. Finite difference methods for seismic wave propagation in heterogeneous materials[J]. Methods in computational physics. 1972;11:1-37.

[15] Kelly KR, Alford RM, Treitel S, et al. Application of finite difference methods to exploration seismology[C]. Topics in numerical analysis, II. 1975: 57-76.

[16] Engquist B, Majda A. Absorbing boundary conditions for numerical simulation of waves[J]. Mathematics of Computation. 1997;31(139):629-651.

[17] Cowsar LC, Dupont TF, Wheeler MF. A priori estimates for mixed finite element approximations of second-order hyperbolic equations with absorbing boundary conditions[J]. SIAM Journal on Numerical Analysis. 1996;33(2):492-504.

[18] Bamberger A, Glowinski R, Tran QH. A domain decomposition method for the acoustic wave equation with discontinuous coefficients and grid change[J]. SIAM Journal on Numerical Analysis. 1996;34(2):603-639.

[19] Lagnese JE, Leugering G. Time-domain decomposition of optimal control problems for the wave equation[J]. Systems & Control Letters. 2003;48(3):229-242.

[20] Ma KY, Sun TJ, Yang DP. Parallel Galerkin domain decomposition procedures for parabolic equation on general domain[J]. Numerical Methods for Partial Differential Equations. 2009;25(5):1167-1194.

[21] Ma KY, Sun TJ. Galerkin domain decomposition procedures for parabolic equations on rectangular domain[J]. International Journal for Numerical Methods in Fluids. 2010;62(4):449-472.

[22] Sun TJ, Ma KY. Dynamic parallel Galerkin domain decomposition procedures with grid modification for parabolic equation[J]. International Journal for Numerical Methods in Fluids. 2011;66(12):1506-1529.

[23] Sun TJ, Ma KY. Domain decomposition procedures combined with H^1-Galerkin mixed finite element method for parabolic equation[J]. Journal of Computational and Applied Mathematics. 2014;267:33-48.

[24] Sun TJ, Ma KY. Parallel Galerkin domain decomposition procedures for wave equation[J]. Journal of Computational and Applied Mathematics. 2010;233(8):1850-1865.

[25] Ma KY, Sun TJ. Parallel Galerkin domain decomposition procedures based on the streamline diffusion method for convection-diffusion problems[J]. Journal of Computational and Applied Mathematics. 2011;235(15):4464-4479.

[26] Adams RA. Sobolev spaces[M]. Academic press; 1975.

[27] Lions JL. Optimal control of systems governed by partial differential equations[M]. Springer-Verlag, Berlin; 1971.

[28] Neittaanmaki P, Tiba D. Optimal control of nonlinear parabolic systems: Theory, algorithms and applications[M]. CRC Press; 1994.

[29] Liu WB, Yan NN. Adaptive finite element methods for optimal control governed by PDEs: C Series in Information and Computational Science 41[M]. Science Press; 2008.

[30] Brenner SC, Scott LR. The mathematical theory of finite element methods, Texts in Applied Mathematics 15[M]. Springer-Verlag, Berlin; 1996.

[31] Zhu LR, Yuan ZQ. Boundary element analysis[M] (in Chinese). Science Press; 2009.

[32] Fu HF, Rui HX. A priori error estimates for optimal control problems governed by transient advection-diffusion equations[J]. Journal of Scientific Computing. 2009;38(3):290-315.

[33] Ciarlet PG. The finite element method for elliptic problems[M]. North-Holland Publisher; 1978.

[34] Nitsche J. L^{∞}-error analysis for finite elements[C]. In the Mathematics of Finite Elements and Applications III (Whiteman,J.R., ed.), Academic Press, New York; 1979;173-186.

[35] Schatz AH, Wahlbin LB. Maximun norm estimates in the finite element method on plane polygonal domains, I[J]. Mathematics of Computation. 1978;32(141):73-109.

[36] Schatz AH, Wahlbin LB. Maximun norm estimates in the finite element method on plane polygonal domains, II[J]. Mathematics of Computation. 1979;33(146):465-492.

[37] Dupont TF. L^2-estimates for Galerkin methods for second order hyperbolic equations [J]. SIAM Journal on Numerical Analysis. 1973;10(5):880-889.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/19835