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The Construction of Implicit One-step Block Hybrid Methods with Multiple Off-grid Points for the Solution of Stiff Differential Equations

Y. Skwame[1], J. Sabo[1*] and T. Y. Kyagya[2]

[1]*Department of Mathematics, Adamawa State University, Mubi, Nigeria.*
[2]*Department of Mathematics and Statistics, Federal University, Wukari, Nigeria.*

Authors' contributions

This work was carried out in collaboration between all authors. Author YS designed the method. Author JS analyzed the basic properties of the method and author TYK implemented the method on some stiff differential equations. All authors read and approved the final manuscript.

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ABSTRACT

This paper is focused on the construction and implementation of implicit one-step hybrid block methods for the solution of stiff ordinary differential equation. The research further investigates the basic properties of implicit one-step block hybrid method. We noticed that the moment the value of an error constant is positive, the order P is odd. And when the value of an error constant is negative then the order P is of even number and the block hybrid method with three off-grid point is of uniform order. The performance of the methods was demonstrated on some stiff initial value problems (IVPs). The result revealed that the hybrid block methods are efficient, accurate and convergent on some stiff ordinary differential equations.

Keywords: Implicit; hybrid block method; BHM; stiff ODEs.

**Corresponding author: E-mail: sabojohn630@yahoo.com;*

1. INTRODUCTION

The general linear methods were introduced to provide a unifying frame work to study consistency, stability and convergence of the traditional methods. More recently, the use hybrid block methods which computes successfully with others like Rang-Kutta and linear multi-step methods, see [1,2,3]. This methods was high order becomes very difficult to drive using inversion algorithm, and another approach has been sought using Maple and Matlab software program.

This paper has been classified into sections. In section 2.0 the MC procedure is constructed involving multiple off-grid collocation points and we analyze on its convergence analysis obtain in a block form. We obtain the order and error constants in a block form in section 3.1, the stability regions are also plotted in section 3.2 a, section 3.3 is the numerical implementation of the block hybrid schemes on stiff (ODEs) and conclusion is given in section 4.0. We consider the numerical solution of first order initial value problems of the form:

$$y' = f(x, y), y(x_0) = y_0 \tag{1.1}$$

where f is continuous and satisfies Lipchitz's condition that guarantees the uniqueness and existence of a solution.

2. CONSTRUCTIONS OF THE METHODS

2.1 Derivation Techniques of MC [1,3,4,5]

Consider the collocation methods defined for the step $|x_n, x_{n+1}|$ by

$$y(x) = \sum_{j=0}^{t-1} \alpha(x) y_{n+1} + h \sum_{j=0}^{m-1} \beta_j(x) f(x_j, y(\bar{x}_j)), \tag{2.1}$$

where t denotes the number of interpolation points x_{n+j}, $j = 0, \cdots, t-1$, and m denotes the number of distinct collocation points $\bar{x}_j \in [x_n, x_{n+k}]$, $j = 0,1,\cdots,m-1$ the points $\bar{x}_j$ are chosen from the step x_{n+j} as well as one or more off-step points.

The following assumptions are made;

1. Although the step size can be variable, for simplicity in our presentation of the analysis in this paper, we assume it is constant $h = x_{n+1} - x_n$, $N = \frac{b-a}{h}$ with the steps given by $\{x_n / x_n = a + nh, n = 0,1,\cdots,N\}$,
2. That (1.1) has a unique solution and the coefficients $\alpha_j(x), \beta_j(x)$ in (2.1) can be represented by polynomial of the form

$$\alpha_j(x) = \sum_{j=0}^{t+m-1} \alpha_{j,i+1} x^i, \quad j\varepsilon(0,1,2,\cdots,t-1) \tag{2.2}$$

$$h\beta_j(x) = h \sum_{j=0}^{t+m-1} \beta_{j,i+1} x^i, \quad j\varepsilon(0,1,2,\cdots,m-1) \tag{2.3}$$

with constant coefficients $\alpha_{j,i+1}, h\beta_{j,i+1}$ and collocation conditions

$$\bar{y}(x_{n+j}) = y_{n+j}, \qquad j\varepsilon\{0,1,\cdots,t-1\} \tag{2.4}$$

$$\bar{y}'(\bar{x}_j) = f(\bar{x}_j, \bar{y}(\bar{x}_j)), \qquad j\varepsilon\{0,1,\cdots m-1\} \tag{2.5}$$

with these assumptions we obtained an MC polynomial in the form

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^i, \quad \alpha_j = \sum_{j=0}^{t-1} C_{i+1,j+1} + \sum_{j=0}^{m-1} C_{i+1,j+1} f_{n+j} \tag{2.6}$$

and also we get D Matrix as follows:

$$\underline{D} = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+2}^2 & \cdots & x_n^{t+m-1} \\ \vdots & & & & \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_0^{t+M-2} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_1^{t+M-2} \\ \vdots & & & & \\ 0 & 1 & 2\bar{x}_{m-1} & \cdots & (t+m-1)\bar{x}_{m-1}^{t+M-2} \end{bmatrix} \tag{2.7}$$

The parameters required for equation (2.7) to obtain one-step block hybrid methods with three off-grid points are $k=1, t=1, m=k+4$: $x=\left\{x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}\right\}$

Using the maple software program and evaluating (2.7) at the grid-points, $x=\left\{x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}\right\}$ we get four members schemes. Hence, the hybrid block methods are as follows:

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{2880} h\left[251 f_n + 646 f_{n+\frac{1}{4}} - 264 f_{n+\frac{1}{2}} + 106 f_{n+\frac{3}{4}} - 19 f_{n+1}\right]$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{360} h\left[29 f_n + 124 f_{n+\frac{1}{4}} + 24 f_{n+\frac{1}{2}} + 4 f_{n+\frac{3}{4}} - f_{n+1}\right]$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{320} h\left[9 f_n + 34 f_{n+\frac{1}{4}} + 24 f_{n+\frac{1}{2}} + 14 f_{n+\frac{3}{4}} - f_{n+1}\right]$$

$$y_{n+1} = y_n + \frac{1}{90} h\left[7 f_n + 32 f_{n+\frac{1}{4}} + 12 f_{n+\frac{1}{2}} + 32 f_{n+\frac{3}{4}} + 7 f_{n+1}\right] \tag{2.8}$$

The parameters required for equation (2.7) to obtain one-step block hybrid method with four off- grid points are $k=1, t=1, m=k+5$: $x=\left\{x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+\frac{7}{8}}, x_{n+1}\right\}$

By using the maple software program and evaluating (2.7) at the grid- points $x=\left\{x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+\frac{7}{8}}, x_{n+1}\right\}$ we get the five members schemes as follows:

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{10080} h\left[811 f_n + 2639 f_{n+\frac{1}{4}} - 1869 f_{n+\frac{1}{2}} + 2261 f_{n+\frac{3}{4}} - 1728 f_{n+\frac{7}{8}} + 406 f_{n+1}\right]$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2520} h\left[193 f_n + 924 f_{n+\frac{1}{4}} + 28 f_{n+\frac{1}{2}} + 308 f_{n+\frac{3}{4}} - 256 f_{n+\frac{7}{8}} + 63 f_{n+1}\right]$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{1120} h\left[29 f_n + 133 f_{n+\frac{1}{4}} + 49 f_{n+\frac{1}{2}} + 119 f_{n+\frac{3}{4}} - 64 f_{n+\frac{7}{8}} + 14 f_{n+1}\right]$$

$$y_{n+\frac{7}{8}} = y_n + \frac{7}{92160} h\left[1021 f_n + 4704 f_{n+\frac{1}{4}} + 1666 f_{n+\frac{1}{2}} + 5096 f_{n+\frac{3}{4}} - 1408 f_{n+\frac{7}{8}} + 441 f_{n+1}\right]$$

$$y_{n+1} = y_n + \frac{1}{90} h\left[7 f_n + 32 f_{n+\frac{1}{4}} + 12 f_{n+\frac{1}{2}} + 32 f_{n+\frac{3}{4}} + 7 f_{n+1}\right] \tag{2.9}$$

2.2 Stability of Block Method

The equations (2.8) when put together formed the block as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{3}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{1}{4}} \end{bmatrix} + h\left\{ \begin{bmatrix} \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{-7}{90} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \right\} + \begin{bmatrix} 0 & 0 & 0 & \frac{251}{2880} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{27}{320} \\ 0 & 0 & 0 & \frac{7}{90} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{1}{4}} \end{bmatrix} \tag{2.10}$$

The characteristic polynomial of the hybrid bock method (2.7) and (2.10) is given as

$$\rho(R) = \det\left[RA^0 - A^1\right], \quad where \quad A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad and \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho(R) = \det\left[R\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right] = \det\begin{pmatrix}R&0&0&-1\\0&R&0&-1\\0&0&R&-1\\0&0&0&R-1\end{pmatrix} = 0$$

Since $= R(R(R(R-1))) \Rightarrow R_1 = 0, R_2 = 0, R_3 = 0, R_4 = 1$

Also, the hybrid block method with four off-grid points, the equations (2.8) when put together formed the block as

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{4}}\\y_{n+\frac{1}{2}}\\y_{n+\frac{3}{4}}\\y_{n+\frac{7}{8}}\\y_{n+1}\end{bmatrix} = \begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{7}{8}}\\y_{n+\frac{3}{4}}\\y_{n+\frac{1}{2}}\\y_{n+\frac{1}{4}}\end{bmatrix} + h\begin{bmatrix}\frac{377}{1440}&\frac{-89}{480}&\frac{323}{1440}&\frac{-6}{35}&\frac{29}{720}\\\frac{11}{30}&\frac{1}{90}&\frac{11}{90}&\frac{-33}{315}&\frac{1}{40}\\\frac{57}{160}&\frac{21}{160}&\frac{51}{160}&\frac{-6}{160}&\frac{3}{90}\\\frac{343}{960}&\frac{583}{46080}&\frac{4459}{11520}&\frac{-77}{720}&\frac{343}{10240}\\\frac{16}{45}&\frac{2}{15}&\frac{16}{45}&0&\frac{7}{90}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{4}}\\f_{n+\frac{1}{2}}\\f_{n+\frac{3}{4}}\\f_{n+\frac{7}{8}}\\f_{n+1}\end{bmatrix} + h\begin{bmatrix}0&0&0&0&\frac{811}{10080}\\0&0&0&0&\frac{193}{2520}\\0&0&0&0&\frac{87}{1120}\\0&0&0&0&\frac{7147}{92160}\\0&0&0&0&\frac{7}{9}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{7}{8}}\\f_{n+\frac{1}{2}}\\f_{n+\frac{3}{4}}\\f_{n+\frac{1}{4}}\end{bmatrix} \quad (2.11)$$

The characteristic of polynomial of the hybrid block method (2.8) and (2.11) is given as

$$\rho(R) = \det\left[RA^0 - A^1\right].$$

where $A^0 = \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$ and $A^1 = \begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}$

Hence, $\det\begin{bmatrix}R&0&0&0&-1\\0&R&0&0&-1\\0&0&R&0&-1\\0&0&0&R&-1\\0&0&0&0&R-1\end{bmatrix} = 0$

$$= R(R(R(R(R-1)))) = 0$$

which implies $R_1 = 0, R_2 = 0, R_3 = 0, R_4 = 0, R_5 = 1$.

$|R_j| \leq 1, \quad j \in \{1, 2, 3, 4, 5\}$ hence the method as a block is zero stable on its own and the hybrid block method is *also* consistant as order $\rho > 1$

3. CONVERGENCE ANALYSIS

3.1 Order and Error Constants of the Hybrid Block Methods

The hybrid block methods which are obtained in a block form with the help of maple software have the following order and error constants for each block hybrid method.

The method $k = 1$ with three off-grid points is of mixed order 5 and 6 and has error constants

$$C_{5,6} = \left[\frac{3}{655360}, \frac{1}{368640}, \frac{3}{655360}, -\frac{1}{1935360}\right]^T$$

The method $k=1$ with four off-grid points is of uniform order 6 and has error constants

$$C_7=[-\tfrac{1159}{1981808640},-\tfrac{53}{123863040},-\tfrac{37}{73400320},-\tfrac{4459}{9059696640},-\tfrac{1}{1935360}]^T$$

In this case, we noticed that the moment the value of an error constant is positive, the order P is odd. And when the value of an error constant is negative then the order P is of even number.

3.2 Regions of Absolute Stability

Using the MATLAB package, we were able to plot the stability regions of the block method (see Figs. 1 and 2). This is done by reformulating the block method as general linear method to obtain the values of the matrices according to [6,7]. These matrices are substituted into the stability matrix and using MATLAB software, the absolute stability regions of the new methods are plotted as shown in Figs. 1 and 2.

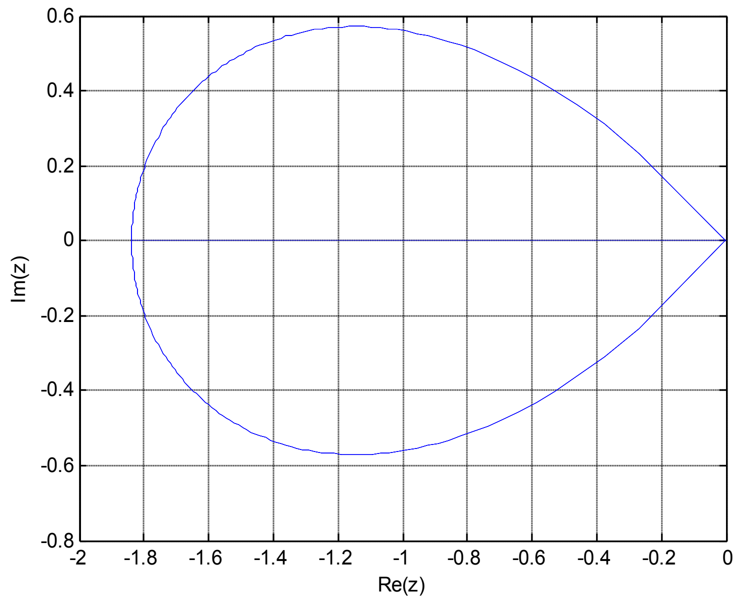

Fig. 1. Stability region of the block hybrid method for *k=1* with three off-grid points

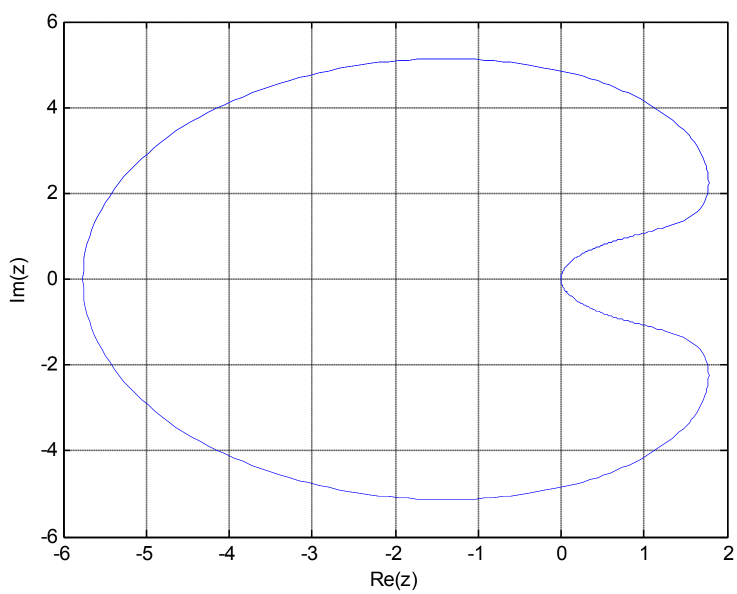

Fig. 2. Stability region of the block hybrid method for *k=1* with four off-grid points

3.3 Numerical Implementation

To study the efficiency of the block hybrid method for $k=1$, we present some numerical examples widely used by several authors such as [3,8].

Experiment 1 $\quad y^1=-10000y, \quad where \quad h=0.1, \quad x\in[0, 0.8]$

$y(x)=e^{-10000x}$

Experiment 2

$$y^1=\begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, \quad y(0)=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad 0 \le x \ge 0.8, \quad h=0.1$$

$$y(x)=\frac{1}{2}\begin{pmatrix} e^{-2x}+e^{-40x}(\cos(40x)+\sin(40x)) \\ e^{-2x}-e^{-40x}(\cos(40x)+\sin(40x)) \\ 2e^{-40x}(\sin(40x)-\cos(40x)) \end{pmatrix}$$

Table 1. Absolute error for experiment 1

Y	$A-stable$ **hybrid block with three off-grid points**	$\alpha(0)-stable$ **hybrid block with four off-grid points**
0.1	9.67×10^{-1}	9.92×10^{-2}
0.2	9.36×10^{-1}	9.84×10^{-3}
0.3	9.05×10^{-1}	9.77×10^{-4}
0.4	8.75×10^{-1}	9.69×10^{-5}
0.5	8.85×10^{-1}	9.62×10^{-6}
0.6	8.19×10^{-1}	9.54×10^{-7}
0.7	7.92×10^{-1}	9.47×10^{-8}
0.8	7.66×10^{-1}	9.39×10^{-9}

Table 2. Absolute error for experiment 2

Y	$A-stable$ **hybrid block with three off-grid points**			$\alpha(0)-stable$ **hybrid block with four off-grid points**		
	y_1	y_2	y_3	y_1	y_2	y_3
0.1	2.98×10^{-2}	2.98×10^{-2}	1.06×10^{-2}	2.23×10^{-2}	2.23×10^{-2}	2.53×10^{-2}
0.2	7.38×10^{-4}	4.03×10^{-4}	1.68×10^{-4}	1.06×10^{-4}	9.14×10^{-5}	1.68×10^{-4}
0.3	1.28×10^{-5}	1.28×10^{-5}	3.76×10^{-5}	8.23×10^{-6}	9.10×10^{-6}	1.33×10^{-5}
0.4	8.08×10^{-8}	8.22×10^{-8}	9.87×10^{-7}	9.60×10^{-6}	9.30×10^{-6}	1.60×10^{-7}
0.5	9.10×10^{-9}	8.30×10^{-9}	2.81×10^{-8}	9.67×10^{-6}	9.67×10^{-6}	1.68×10^{-9}
0.6	2.00×10^{-10}	1.10×10^{-9}	4.42×10^{-10}	9.50×10^{-6}	9.50×10^{-6}	9.12×10^{-11}
0.7	4.00×10^{-10}	6.00×10^{-10}	6.98×10^{-11}	9.08×10^{-6}	9.08×10^{-6}	1.05×10^{-10}
0.8	3.00×10^{-10}	5.00×10^{-10}	9.99×10^{-11}	8.49×10^{-6}	8.49×10^{-6}	6.84×10^{-11}

It is obvious from the results presented in Tables 1 and 2 that the four off-grid performs better than the methods with three off-grid points.

4. CONCLUSIONS

It is evident from Tables 1 and 2 that our proposed methods are indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the methods, with three off-grid point is $A-stable$ and that with four off-grid points is $\alpha(0)-stable$. Comparing the two schemes derived, the $\alpha(0)-stable$ hybrid methods performs better than the $A-stable$ hybrid methods, in view of the results presented in the Tables 1 and 2.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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