



## Two Step Trigonometrically Fitted Method for Numerical Solution of Initial Value Problems with Oscillating Solutions

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### Authors' contributions

This work was carried out in collaboration between both authors. Author OOO proposed the algorithms. Author AOA developed, analysed and implemented the method. Authors AOA and OOO drafted the manuscripts. Both authors read and approved the final manuscript.

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## Abstract

A continuous two-step method using trigonometric function basis is developed and used to produce two discrete methods which are simultaneously applied as numerical integrators by assembling them into a block method with trigonometric basis for solving oscillatory initial value problems (IVP). The stability property of the method is well discussed and the performance of the method is demonstrated on some numerical examples to show accuracy and efficiency advantages.

*Keywords:* Two-step block method; initial value problems; Trigonometrically fitted method; first order system.

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## 1 Introduction

This paper considered the class of second order differential equation of the form

$$y' = f(x, y), \quad y(a) = y_0, \quad x \in [a, b] \quad (1.1)$$

with oscillatory solutions where  $f : \mathfrak{R} \times \mathfrak{R}^m \rightarrow \mathfrak{R}^m$ ,  $y, y_0 \in \mathfrak{R}^m$

In the field of ordinary differential equations, a nontrivial solution to an ordinary differential equation

$$F(x, y, y', \dots, y^{(n-1)}) = y^{(n)}$$

is called oscillating if it has an infinite number of roots. The differential equation is oscillating if it has an oscillating solution. The number of roots carries information on the spectrum of the associated boundary value problems. For example, the differential equation  $y'' + y = 0$  is oscillating as  $\sin x$  is a solution.

Oscillatory Initial Value Problems usually occur in area such as Quantum Mechanics, Biological Sciences, Classical Mechanics and Celestial Mechanics. A lot of numerical methods based on polynomial and non polynomial basis have been developed for solving this important class of problems [1, 2, 3, 4, 5, 6, 7, 8].

This paper constructs two-step method with trigonometric basis, which provides two discrete methods that are combined and applied as block two step method with trigonometric basis which take the frequency of the solution as a prior knowledge.

This paper adopts the approach given in Ngwane and Jator [8] and Jator et al. [7], where the continuous two-step method with trigonometric basis is used to generate the main and one additional method which are combined and used as a two-step block method to simultaneously produce approximations.

$$\{y_{n+1}, y_{n+2}\} \text{ at block of points } \{x_{n+1}, x_{n+2}\},$$

$h = x_{n+1} - x_n$ ,  $n = 0, \dots, N - 1$  on a partition  $[a, b]$ , where  $h$  is the step size,  $n$  is the grid index and  $N > 0$  is the number of steps. The block methods generates approximations  $\{y_{n+1}, y_{n+2}\}$  to the exact solution  $\{y(x_{n+1}), y(x_{n+2})\}$

The paper is organized as follows. In section 2, a trigonometric basis representation  $U(x)$  for the exact solution  $y(x)$  which are used to generate two discrete methods and which are also combined to solve (1.1) is derived. The analysis and implementation of the new method are given in section 3. Numerical examples are given in section 4 to show accuracy and efficiency of the new block method. The conclusion of the paper is discussed in section 5.

## 2 Derivation of the Method

A two-step with trigonometric basis which produces two discrete methods as by-products is derived. The main method has the form

$$y_{n+1} = y_n + h\{\alpha_0(u)f_n + \alpha_1(u)f_{n+1} + \alpha_2(u)f_{n+2}\} \quad (2.1)$$

and the additional method is given by

$$y_{n+2} = y_n + h\{\hat{\alpha}_0(u)f_n + \hat{\alpha}_1(u)f_{n+1} + \hat{\alpha}_2(u)f_{n+2}\} \quad (2.2)$$

where  $u = wh$ ,  $\alpha_j$ ,  $\hat{\alpha}_1$ ,  $j = 0(1)2$ , are coefficients that depend on the step-size and frequency.  $y_{n+1}$  is the numerical approximation to analytical solution  $y(x_{n+1})$ .

In order to obtain (2.1) and (2.2), the paper proceeds by seeking to approximate the exact solution  $y(x)$  on the interval  $[x_n, x_{n+h}]$  through interpolation function  $U(x)$  of the form:

$$U(x) = a_0 + a_1x + a_2 \sin(wx) + a_3 \cos(wx), \tag{2.3}$$

where  $a_i$ ,  $i = 0(1)3$ , are coefficients that must be uniquely determined. The paper then imposes that the interpolating function (2.3) coincides with the analytical solution at the end point  $x_n$  to obtain the equation

$$U(x_n) = y_n, \tag{2.4}$$

It is also demanded that the function (2.3) satisfies the differential equation (1.1) at points  $x_{n+j}$ ,  $j = 0, 1, 2$  to obtain the following set of three equations:

$$U'(x_n) = f_n, \quad U'(x_{n+1}) = f_{n+1}, \quad U'(x_{n+2}) = f_{n+2} \tag{2.5}$$

The system of equation in (2.4) and (2.5) are solved by "Cramer's rule" to obtain  $a_j$ ,  $j = 0(1)3$ . Continuous two step method with trigonometric basis is constructed by substituting the values of  $a_j$  into (2.3). After some algebraic manipulation, the new two-step method is of the form:

$$U(x) = y_n + h\{\alpha_0(w, x)f_n + \alpha_1(w, x)f_{n+1} + \alpha_2(w, x)f_{n+2}\}, \tag{2.6}$$

where  $w$  is the frequency,  $\alpha_j(w, x)$ ,  $j = 0(1)2$  are continuous coefficients. The continuous method (2.6) is used to generate the main method of the form (2.1) and an additional method of the form (2.2) by evaluating (2.6) at  $x = \{x_{n+1}, x_{n+2}\}$  and letting  $u = wh$ , the coefficients of (2.1) and (2.2) obtained are obtained

$$\alpha_0 = \frac{1}{2} \frac{(-u \sin(u) - 2(\cos(u))^2 + 2 \cos(u))}{u \sin(u) (\cos(u) - 1)}$$

$$\alpha_1 = \frac{1}{2} \frac{(2u \sin(u) \cos(u) + 2(\cos(u))^2 - 2)}{u \sin(u) (\cos(u) - 1)} \tag{2.7}$$

$$\alpha_2 = \frac{1}{2} \frac{(-u \sin(u) - 2 \cos(u) + 2)}{u \sin(u) (\cos(u) - 1)}$$

$$\hat{\alpha}_0 = \frac{(-u + \sin(u))}{u (\cos(u) - 1)}$$

$$\hat{\alpha}_1 = \frac{(2u \cos(u) - 2 \sin(u))}{u (\cos(u) - 1)} \tag{2.8}$$

$$\hat{\alpha}_2 = \frac{(-u + \sin(u))}{u (\cos(u) - 1)}$$

### 3 Error Analysis and Stability

#### 3.1 Local truncation error

Taylor series is used for small values of  $u$  (see Simos [9]). The coefficient of (2.7) and (2.8) can be expressed as:

$$\alpha_0 = \frac{5}{12} + \frac{19u^2}{720} + \frac{23u^4}{10080} + \frac{263u^6}{1209600} + \frac{1033u^8}{47900160} + \frac{945979u^{10}}{435891456000} + \dots$$

$$\begin{aligned} \alpha_1 &= \frac{2}{3} - \frac{u^2}{90} - \frac{u^4}{2520} - \frac{u^6}{75600} - \frac{u^8}{2395008} - \frac{691 u^{10}}{54486432000} + \dots \quad (3.1) \\ \alpha_2 &= -\frac{1}{12} - \frac{11 u^2}{720} - \frac{19 u^4}{10080} - \frac{247 u^6}{1209600} - \frac{1013 u^8}{47900160} - \frac{940451 u^{10}}{435891456000} + \dots \\ \hat{\alpha}_0 &= \frac{1}{3} + \frac{u^2}{90} + \frac{u^4}{2520} + \frac{u^6}{75600} + \frac{u^8}{2395008} + \frac{691 u^{10}}{54486432000} + \dots \\ \hat{\alpha}_1 &= \frac{4}{3} - \frac{u^2}{45} - \frac{u^4}{1260} - \frac{u^6}{37800} - \frac{u^8}{1197504} - \frac{691 u^{10}}{27243216000} + \dots \quad (3.2) \\ \hat{\alpha}_2 &= \frac{1}{3} + \frac{u^2}{90} + \frac{u^4}{2520} + \frac{u^6}{75600} + \frac{u^8}{2395008} + \frac{691 u^{10}}{54486432000} + \dots \end{aligned}$$

The Local Truncation Error(LTE) for methods (2.1) and (2.2) are given by

$$\begin{aligned} \text{LTE}(2) &= \frac{h^4}{24} \left( w^2 y^{(2)}(x_n) + y^{(4)}(x_n) \right) \\ \text{LTE}(3) &= -\frac{h^5}{90} \left( w^2 y^{(2)}(x_n) + y^{(4)}(x_n) \right) \quad (3.3) \end{aligned}$$

### 3.2 Stability

The block by block method for computing vectors  $Y_0, Y_1, Y_2, \dots$  in sequence is defined (see Fatunla [10]). Let the  $\eta$ -vector( $\eta = 2$  is the number of points within the block)  $Y_\mu, Y_{\mu-1}, F_\mu$  and  $F_{\mu-1}$  be given as  $Y_\mu = (y_{n+1}, y_{n+2})^T, Y_{\mu-1} = (y_{n-1}, y_n)^T, F_\mu = (f_{n+1}, f_{n+2})^T, F_{\mu-1} = (f_{n-1}, f_n)^T$ , then the 1-block 2-point method for (1.1) is given as:

$$Y_\mu = \sum_{i=1}^1 A^{(i)} Y_{\mu-1} + \sum_{i=0}^1 B^{(i)} F_{\mu-1}, \quad (3.4)$$

where  $A^{(i)}, B^{(i)}, i = 0, 1$  are  $2 \times 2$  matrices whose entries are given by the coefficient of (2.1) and (2.2).

#### Zero stability

The block method in (3.4) is zero stable provided the roots  $R_j, j = 1, 2$  of the first characteristic polynomial  $\rho(R)$  is specified by

$$\rho(R) = \det \left[ \sum_{i=0}^1 A^{(i)} R^{i-1} \right] = 0, \quad A^{(0)} = -I \quad (3.5)$$

satisfies  $|R_j| \leq 1, j = 1, 2$  and for those roots with  $|R_j| = 1$ , the multiplicity does not exceed 1 (see Fatunla [10]).

#### Consistency

Our block method in (3.4) is consistent(it has order  $p > 1$ ).  
The block method is convergent(Convergence = Zero stability+ consistency)

## 4 Implementation of the Scheme

Method (2.1) and (2.2) are implemented more efficiently as simultaneous integrator for IVPs without requiring starting values and predictors. This paper proceeds by explicitly obtaining initial

conditions at  $x_{n+2}$ ,  $n = 0, 2, \dots, N - 2$  using the computed values  $y(x_{n+2})=y_{n+2}$  over sub-intervals  $[x_0, x_2], \dots, [x_{N-2}, x_N]$ . For instance, using equations (2.1 – 2.2), and with  $n = 0$ .  $(y_1, y_2)^T$ , are simultaneously obtained over the sub-interval  $[x_0, x_2]$ , as  $y_0$  is known from the IVP (1). Also for  $n = 2$ ,  $(y_3, y_4)^T$  are simultaneously obtained over the sub-interval  $[x_2, x_4]$ , as  $y_2$  is known from the previous block, where  $T$  is the transpose and so on. Hence, the sub-interval do not over-lap and the solution in this manner is more accurate than those obtained in the conventional fashion.

On the choice of frequency( $w$ ), the method in Vigo-Aguiar and Ramos [11], on their paper titled " On the choice of the frequency in trigonometrically-fitted method" is adopted. The paper uses the trigonometrically-fitted method to obtain an approximate solution to some nonlinear oscillators and also presents a strategy for the choice of frequency in trigonometrically-fitted method.

## 5 Numerical Examples

In order to study the accuracy and efficiency of the developed methods, This paper presents some numerical experiments:

### Example 1.1

$y'' = -100y + 99\sin(x)$ ,  $y(0) = 1$ ,  $y'(0) = 11$  (Adeniran et al. [2])  
*Exact solution* : $y(x) = \cos(10x) + \sin(10x) + \sin(x)$

**Table 1. Showing the exact solutions, computed results and error from the proposed methods for example 1.1 with  $h = \frac{1}{320}$ ,  $w = 1$**

x	yExact	yComputed	error	Error in Adeniran et al. [2]
$\frac{1}{320}$	1.03388166738420	1.03388166738410	$9.99 \times 10^{-14}$	$9.170 \times 10^{-11}$
$\frac{2}{320}$	1.06675678785246	1.06675678785234	$1.20 \times 10^{-13}$	-
$\frac{3}{320}$	1.09859628036501	1.09859628036580	$7.90 \times 10^{-13}$	$3.0905 \times 10^{-10}$
$\frac{4}{320}$	1.12937207509627	1.12937207509610	$1.69 \times 10^{-13}$	-
$\frac{5}{320}$	1.15905714081491	1.15905714081441	$5.00 \times 10^{-13}$	-
$\frac{6}{320}$	1.18762550988244	1.18762550988224	$2.00 \times 10^{-13}$	$4.8987 \times 10^{-10}$
$\frac{7}{320}$	1.21505230844501	1.21505230844510	$8.99 \times 10^{-14}$	-
$\frac{8}{320}$	1.24131377434580	1.24131377434560	$2.00 \times 10^{-13}$	-
$\frac{9}{320}$	1.26638728387076	1.1.26638728387046	$3.00 \times 10^{-13}$	-
$\frac{10}{320}$	1.29025137290459	1.29025137290430	$2.99 \times 10^{-13}$	-

The numerical result for Example 1.1 were presented in Tables 1. The problem was compared to other existing method. The new two step trigonometric method displayed better accuracy within the range of integration.

### Example(1.2)

The initial value problem is considered

$$y''(x) = \frac{(y')^2}{2y} - 2y, \quad y\left(\frac{\pi}{6}\right) = \frac{1}{4} \quad \text{and} \quad y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad (\text{Alabi et al. [12]})$$

exact solution:  $y = \sin^2 x$

**Table 2. Showing the exact solutions, computed results and error from the proposed methods for Problem 1.2,  $h = 0.1$ ,  $w = 1$ .**

x	yExact	yComputed	Error	Error in Alabi et al. [12]
0.1	0.0099667110793792	0.00996671107827122	$1.1 \times 10^{-12}$	
0.2	0.0394695029985574	0.0394695029786944	$2.0 \times 10^{-11}$	
0.3	0.0873321925451611	0.0873321924526160	$9.3 \times 10^{-11}$	
0.4	0.1516466453264171	0.151646645281091	$4.5 \times 10^{-11}$	
0.5	0.229848847065930	0.229848847017083	$4.9 \times 10^{-11}$	
0.6	0.318821122761663	0.318821122673452	$8.8 \times 10^{-11}$	$1.013 \times 10^{-08}$
0.7	0.415016428549879	0.415016428485594	$6.4 \times 10^{-11}$	$4.782 \times 10^{-08}$
0.8	0.514599761150645	0.514599761134647	$1.6 \times 10^{-11}$	$1.109 \times 10^{-07}$
0.9	0.613601047346543	0.613601047330533	$1.6 \times 10^{-10}$	$1.892 \times 10^{-07}$
1.0	0.708073418273572	0.708073418262838	$1.1 \times 10^{-10}$	$1.196 \times 10^{-07}$
1.1	0.794250558627672	0.794250558333421	$2.9 \times 10^{-10}$	$3.019 \times 10^{-07}$
1.2	0.868696857770622	0.868696848083653	$9.7 \times 10^{-09}$	$2.561 \times 10^{-07}$
1.3	0.928444376684474	0.928444372840036	$3.8 \times 10^{-09}$	$1.435 \times 10^{-07}$
1.4	0.971111170334329	0.971111163223212	$7.1 \times 10^{-09}$	$1.019 \times 10^{-07}$
1.5	0.994996248300222	0.994996243303974	$5.0 \times 10^{-09}$	$2.319 \times 10^{-07}$
1.6	0.999147387897376	0.999147386423497	$1.5 \times 10^{-09}$	$5.892 \times 10^{-07}$
1.7	0.983399096289731	0.983399092298768	$4.0 \times 10^{-09}$	$1.013 \times 10^{-06}$
1.8	0.948379208167073	0.948379206787865	$1.4 \times 10^{-09}$	$1.211 \times 10^{-06}$
1.9	0.895483855957207	0.895483801118647	$5.5 \times 10^{-08}$	$1.287 \times 10^{-06}$
2.0	0.826821810431807	0.826821783609997	$2.7 \times 10^{-08}$	$1.435 \times 10^{-06}$

The numerical result for Example 1.2 were presented in Tables 2. The problem was compared to other existing method. The new two step trigonometric method displayed better accuracy within the range of integration.

## 6 Conclusion

The two-step trigonometrically fitted method for solving oscillatory IVPs generated in this paper is accurate, efficient, consistent and zero stable. This method is self-starting and requires no predictor it requires only two functions evaluation at each integration step. The method complete favorably with other existing methods [12, 2] in the literature.

## Competing Interests

Authors have declared that no competing interests exist.

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