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On a Question of Prime Labeling of Graphs

A. M. C. U. M. Athapattu[1*] and P. G. R. S. Ranasinghe[1]

[1]Department of Mathematics, Faculty of Science, University of Peradeniya, Peradeniya, Sri Lanka.

Authors' contributions

This work was carried out in collaboration between both authors. Authors AMCUMA and PGRSR designed the study, performed the statistical analysis, wrote the protocol, and managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In the field of graph theory, the complete graph K_n of n vertices is a simple undirected graph such that every pair of distinct vertices is connected by a unique edge. In the present work, we introduce planar subgraph PG_n of K_n with maximal number of edges $3(n-2)$. Generally, PG_n does not admit prime labeling. We present an algorithm to obtain prime-labeled subgraphs of PG_n. We conclude the paper by stating two conjectures based on labeling of PG_n. In particular, the planar subgraph admits anti-magic labeling but does not admit edge magic total labeling.

Keywords: Planar graph; prime labeling; anti-magic labeling; edge magic total labeling.

1 Introduction

Graph theory has always been an exciting area of research opening up to many avenues. One of the key branches in it is Graph labelling which has its origins traced back to the 19th century when the British mathematician Arthur Cayley proved that there are n^{n-2} distinct labeled trees of order n (Cayley's Tree Formula), where n is the number of vertices of the tree [1]. Most graph labellings traced their origin to labellings presented by Alexander Rosa. In Rosa A [2], he identified three types of labellings, which he

**Corresponding author: E-mail: chathurikaa@sci.pdn.ac.lk;*

called $\alpha -$, $\beta -$, and $\rho -$labellings. The $\beta -$labellings were later renamed as "graceful" by Solomon Golomb [3], and since then the name has been occupied in the literature.

Graph labellings provide useful mathematical models for a wide range of applications, such as data security, coding theory, and communication networks. More detailed discussions graph labelling and applications can be found in Joseph Gallian's survey article [4] and the related references therein.

Let $G = (V, E)$ be a graph, where $V \equiv V(G)$ is the vertex set and $E \equiv E(G)$ is the edge set. If there are n vertices in G, by *a* prime labelling *of G*, we mean *a* vertex labelling of G with the distinct integers in the set $\{1, 2, \dots, n\}$ such that the labels of every two adjacent vertices of G are relatively prime. More precisely, a prime labelling is a bijective function $f: V(G) \to \{1, 2, \dots, n\}$ such that $\gcd(f(u), f(v)) = 1$ for each pair u, v of adjacent vertices of G. If there exists a prime labelling of G, then G is called a prime graph. This concept was originated by Roger Entringer see [4] around 1980.

Magic and antimagic labellings are among the oldest labelling schemes in graph theory where the former is introduced by J. Sedláček [5] in relation to the notion of magic squares. In answering a question raised by Sedláček, B. M. Stewart [6,7] studied various ways of carrying out edge-labelling. He refers a connected graph as semi-magic if there is an edge-labelling such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. A *semi-magic* labelling where the edges are labeled with distinct positive integers is called a magic labelling. In 1978, Claude Berge [8] introduced the notion *regularisable* for semi-magic. A magic labelling is called *super magic* if the set of edge labels consists of consecutive positive integers.

We shall now provide precise mathematical statements. A labelling is called a *magic labelling* of a graph G if it is an injective mapping from the edge set E into the set of positive integers such that the sum of the labels of all the edges incident to a given vertex is independent of this vertex. Such a graph is called *magic* [9]. A graph with $|E(G)|$ number of edges is called anti magic if its edges can be labeled with $1, 2, \dots, |E(G)|$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct [9]. A magic labelling is called *super magic* if the set of all the labels of the edges $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is super magic if and only if there exists a super magic labeling of G [9].

Our work is motivated by Baskar [10], where he investigated the maximum possible edges that can be constructed in a graph with n vertices having the prime labeling property. Taking into account the multiplicative number-theoretic Euler totient-function or Euler phi-function $\phi(\cdot)$, he showed that a maximal number of edges in a simple vertex prime labelling graph with n vertices is $\sum_{k=2}^{n} \phi(k)$. Moreover, he established that the sum is bounded below by $3\,(n-2)$ and bounded above by $(n-1)/2$. For a positive integer n, the *Euler phi-function* $\phi(n)$ is defined to be the number of positive integers not exceeding n that are relatively prime to n [11], p. 234). The objective of the preparation of our paper is to make Baskar's arguments precise and present the properties of simple planar subgraph of complete graph. In the following sections we present the results related to the planar subgraph of complete graphs.

2 Planar Subgraph of Complete Graph

Let $G = (V, E)$ be a graph with n vertices so that $V = \{v_1, v_2, \dots, v_n\}$ and let e_k be a typical element in E which can be identified with an unordered pair (v_i, v_j) of vertices. As an example, consider the complete graph with five vertices, K_5 with $V = \{1,2,3,4,5\}$ and $E = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$. In particular, $E = \{(a, b) | (a, b) = (b, a)\ \forall\, a, b \in \mathbb{Z}_n\}$ gives the collection of edges of complete graph; K_n for $n \in \mathbb{Z}^+$. Moreover, on removal of the set of edges from K_n, we get a planar graph. This new class of planar subgraphs of K_n is denoted by PG_n for $n \geq 5$. Here we give the proper definition for PG_n.

Definition: The graph $PG_n = (V, E)$ with $V = \{1,2,3, \dots, n\}$ and $E = E(K_n)\setminus\{(k, l) \mid 2 \leq k \leq n - 3 \; and \; k + 2 \leq l \leq n - 1\}$ is a planar graph.

Since the number of edges of PG_n for $n \geq 5$ is $3(n - 2)$, the graph PG_n is a simple planar subgraph of K_n which has the maximum number of edges. The pseudo-code of obtaining PG_n is as follows:

Algorithm I

Begin
Number n
Step 1 : Output "Enter the number of vertices"
Input n
For counter_1 $= 1$ to $n - 2$
Step 2 :If (counter_1 $= 1$), then
For counter_2 $= 2$ to n
Output (counter_1, counter_2)
End For
Else
Step 3 :Output (counter_1, counter_1 $+ 1$)
Output (counter_1, n)
End If
End For
Step 4 : Output $(n - 1, n)$
End

Fig. 1. Illustrates the view of the planar graph PG_{14} which was obtained from the above algorithm

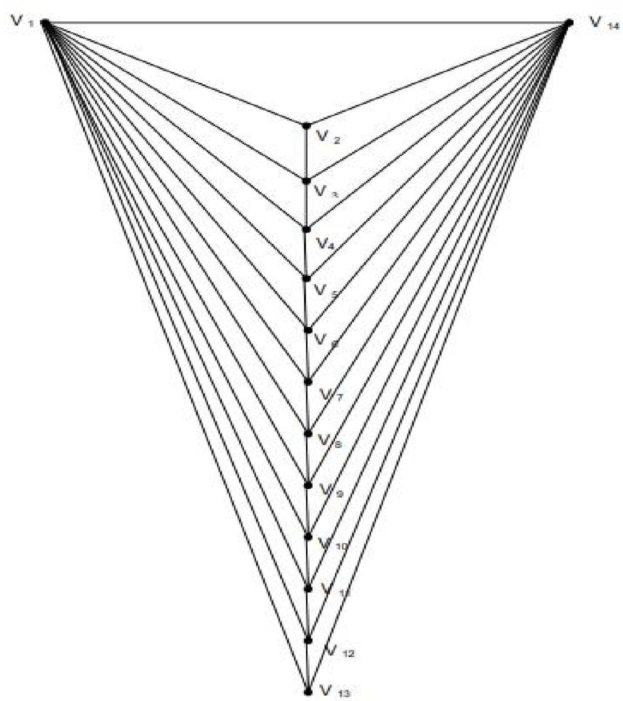

Fig. 1. The graph of PG_{14}

Generally, the graph PG_n does not admit prime labeling. For an instance, PG_6 is not a prime graph. But after removing few edges from PG_n, we obtain a prime subgraph of K_n. Fig. 2 illustrates the prime graph obtained in algorithm II.

Algorithm II

Begin
Number n
Step 1 : Output "Enter the number of vertices"
    Input n
Step 2 : For $k = 1$ to $n - 1$
Step 3 : For $l = k + 1$ to n
Step 4 :If $\gcd(k, l) = 1$, then $i = k$ and $j = l$.
Step 5 : If $2 \leq i \leq n - 3$ and $i + 2 \leq l \leq n - 1$, then
    Output "(i, j) is not an edge"
    Else
    Output (i, j)
    Else
    Output "k and l are not relatively prime"
    End For
    End For
End

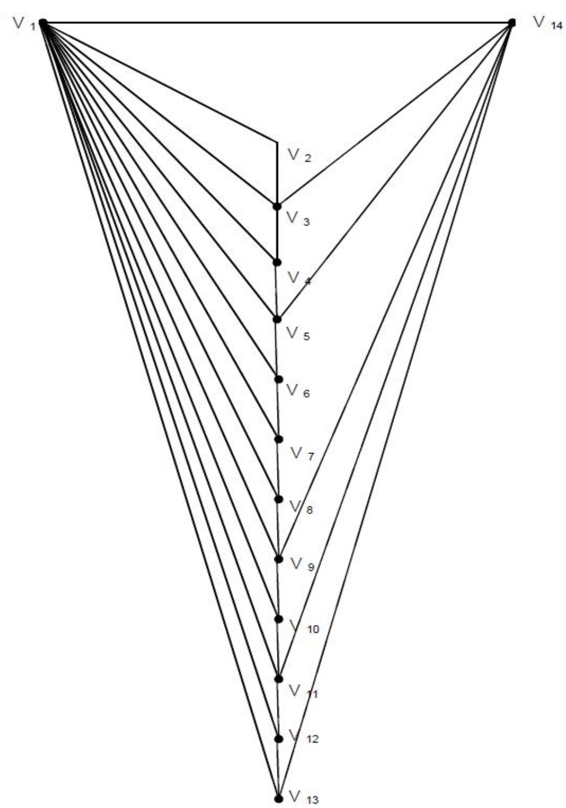

Fig. 2. The prime graph resulting from Algorithm II

3 Labeling of Planar Subgraph of K_n

In this section we present two types of labeling. The graph PG_n admits anti-magic labeling but not edge-magic labeling. Here we recall definition of anti-magic graph. For a connected graph G with e number of edges, an anti-magic edge labeling is a map,

$$\lambda : E(G) \rightarrow \{1,2,3,\ldots,e\},$$

such that the induced vertex sum $f: E(G) \rightarrow \mathbb{Z}^+$ given by $f(u) = \sum\{f(uv) : E(G)\}$ is injective. A graph is called anti-magic if it admits an anti-magic labeling.

In particular, the weights of any two vertices are not equal. This labeling can be done in cyclic form as in Fig. 3.

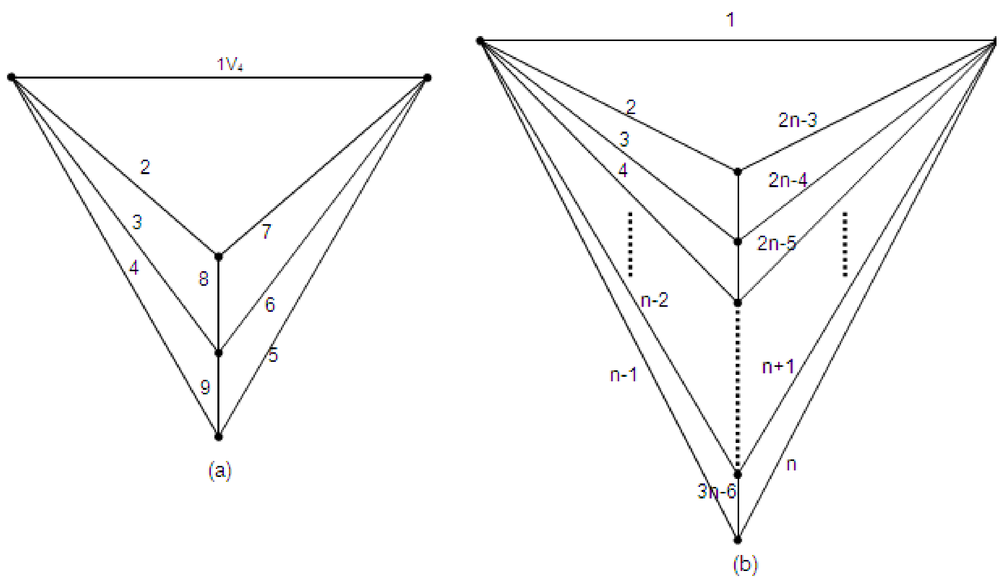

Fig. 3. (a) The anti-magic labeling of PG_5 and (b) The anti-magic labeling of PG_n

An edge-magic total labeling on a graph G is a one-to- one map λ from $V(G) \cup E(G)$ onto the integers $1,2,\ldots,v+e$, where $v = |V(G)|$ and $e = |E(G)|$, with the property that, given edge(x,y),

$\lambda(x) + \lambda(x,y) + \lambda(y) = k$ for some constant k.

Here k is called the magic sum of G. Any graph with an edge-magic total labeling is called edge-magic.

In Marr AM et al. [12], the possible values for the magic sum k and the corresponding sum of vertex labels are listed. Accordingly, we have,

$$\binom{v+1}{2} \leq s \leq ve + \binom{v+1}{2} \text{ and,}$$

$$k = \frac{v(v+1)(v^2+v+2)+8(v-2)s}{4v(v-1)}.$$

With these results we are able to obtain values for s and r which are necessary to apply edge-magic total labeling for the planar graph PG_n. For an example, when $v = 5$, the possibilities are $s = 20, 30, 40, 50, 60$ and $k = 18, 21, 24, 27, 30$. Fig. 4 is the edge-magic total labeling of planar graph PG_5 when $k = 18, 24$ and $s = 20,\ 40$.

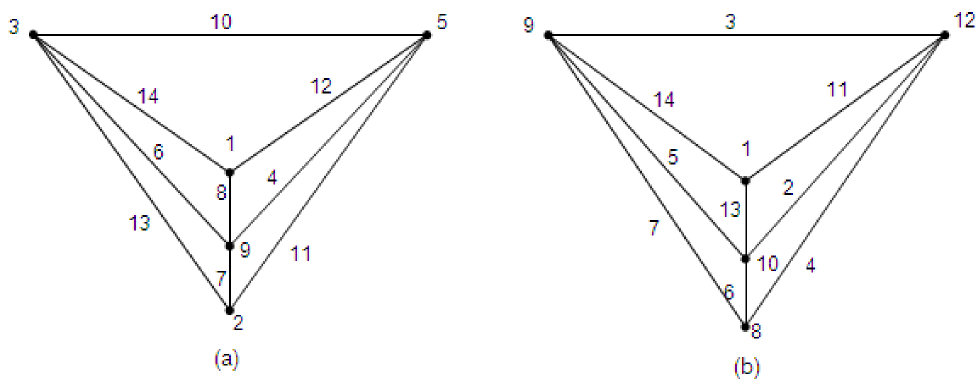

Fig. 4. (a) When $s = 20$, $k = 18$ and (b) when $s = 40$, $k = 24$

As one of the future directions of extending our results, we briefly comment on *Cordial labeling.* In 1987, I. Cahit introduced the notion of Cordial labeling of graphs; Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Then is called a *cordial labeling* of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1 Cahit I [13]. In 1990, Cahit established numerous results on cordial graph labeling, such as "every tree is cordial"; "K_n is cordial if and only if $n \leq 3$"; and "$K_{m,n}$ is cordial for all m and n Cahit I [14]. In 2016, R. Ponraj introduced the concept of k-prime cordial labeling of graphs [15]. The map $f: V(G) \longrightarrow \{1,2,\ldots,k\}$ is called k-prime G if $\left|v_f(i) - v_f(j)\right| \leq 1,\ i,j \in \{1,2,\ldots,k\}$ and $\left|e_f(0) - e_f(1)\right| \leq 1$. Here $v_f(a)$ denotes the number of vertices labeled with a, $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and 0, respectively. Moreover, K_n is 4-prime cordial if and only if $n \leq 3$, Cahit I [14]. This idea motivated to see cordiality of the subgraph PG_n of K_n. In particularly, it is an open problem whether PG_n is 4-prime cordial or not.

4 Conclusion

In the foregoing sections, we have introduced the planar subgraph PG_n of the complete graph K_n with maximum number of edges. In general, the graph PG_n is not a prime graph. We have presented two algorithms which give sketch of the planar subgraph PG_n and prime subgraph of PG_n respectively. In conclusion, we state couple of conjectures related to the current work.

Conjecture 1: The planar graph PG_n is anti-magic.

Conjecture 2: The planar graph PG_n does not have an edge-magic total labeling when $n > 5$.

Competing Interests

Authors have declared that no competing interests exist.

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