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On Rings where Certain Subsets are Multiplicatively Generated by Idempotents

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In a Boolean ring, every element is trivially multiplicatively generated by idempotent elements. In this paper, we study the structure of rings in which certain subsets are multiplicatively generated by idempotents or multiplicatively generated by idempotents and nilpotents.

Keywords: Jacobson radical; commutator ideal; artinian ring; semisimple ring.

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1 Introduction

In a Boolean ring every element is trivially a product of idempotents. This naturally motivates the study of the structure of a ring, which as a semi-group, is generated by its idempotents. That is, rings which are multiplicatively generated by idempotents. This also motivated the study of rings which are multiplicatively generated by idempotents and nilpotents in [1]. Theorems A and B below were proved by Putcha and Yaqub [2], and Theorem C below was proved by Abu-Khuzam in [1].

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Theorem A: Let R be a ring which is multiplicatively generated by idempotents. If R has an identity element, then R is Boolean.

Theorem B: Let R be a ring which is multiplicatively generated by idempotents. If R is finite, then R is Boolean.

Theorem C: Let R be a ring which is multiplicatively generated by idempotents and nilpotents such that the set N of nilpotent elements is commutative, then N forms an ideal of R and R/N is Boolean.

In this paper, we have tried to generalize the above results by assuming that only certain subsets of R, such as the set of non-nilpotent elements (R\N), or the set of elements that are not in the Jacobson radical (R\J), or the set of non-central elements (R\C), are multiplicatively generated by idempotents. The conclusion in each of Theorems 1, 2, 3, and 4 for the ring is the next best structure to being Boolean.

2 Main Results

Throughout, R is a ring, N is the set of nilpotents, C is the center, J is the Jacobson radical of R, C(R) is the commutator ideal of R, and Z is the ring of integers. As usual [x,y] will denote the commutator xy-yx.

Theorem 1. Let R be a finite ring such that

(1) The set R\N is multiplicatively generated by idempotents.

Then N is an ideal of R and R/N is Boolean.

The following Lemmas are needed for the proof of Theorem 1 .

Lemma 1. Any homomorphic image of a ring satisfying (1) must satisfy (1).

Proof. Let $\phi: R \to R'$ be a surjective ring homomorphism. Let $x' = \phi(x)$ be any element of R'. Using (1),

$x \in N \;\; or \;\; x = e_1.e_2...e_n$, where e_i is idempotent in R. Thus,

$x^n = 0 \quad or \quad x = e_1.e_2....e_n$. So,

$x'^n = \phi(x^n) = 0 \quad or \quad x' = \phi(x) = \phi(e_1)\phi(e_2)...\varphi(e_n)$.

Clearly, $\phi(e_i)^2 = \phi(e_i^2) = \phi(e_i)$. So R' satisfies (1).

The following Lemma is well known:

Lemma 2. Let R be a ring. If the commutator ideal C(R) of R is nil, then the set N of nilpotents is an ideal of R.

Lemma 3. Let R be a ring with identity such that R satisfies (1). Then R is Boolean.

Proof:

Let a $\in$ N. Then a+1 is invertible and a+1 $\notin$ N. Hence, by (1),

$1 + a = e_1. e_2. \dots e_k$, e_i is idempotent.

Multiplying both sides by e_1 from the left, we get

$$e_1 (1 + a) = e_1. e_2 \dots e_k , \quad \text{since } e_1 \text{ is idempotent}$$
$$= 1 + a.$$

But 1 + a has an inverse in R, hence $e_1 = 1$.

Similarly, we show that $e_2 = \dots = e_k = 1$.

Hence 1 + a = 1 and a = 0. So N=0, and hence R is multiplicatively generated by idempotents. So, to show that R is a Boolean ring, it is enough to prove that the idempotents are central.

Let e be an idempotent element in R and let a = ex – exe. Clearly,

$$a^2 = (ex - exe)(ex - exe) = 0.$$

So a = 0. Therefore, ex = exe for every $x \in R$.

Following the same above argument, we consider a = xe – exe and we show that xe = exe for every x in R. So, ex = xe for all x in R.

Thus e is central and R is Boolean.

Proof of Theorem 1:

Let R be a finite ring satisfying (1).

Case 1. If J=0, then R is a finite semisimple Artinian ring with identity satisfying (1). So R is Boolean, by Lemma 3, and the theorem follows.

Case 2. If $J \neq 0$, then R/J is a semisimple Artinian ring with identity satisfying (1), by Lemma 1. So by Lemma 3, R/J is Boolean and hence commutative. Thus, xy-yx ε J for all x,y in R. But J is nil, since R is finite (hence Artinian). Hence,

$$xy-yx \in N \text{ for all } x, y \text{ in } R.$$

Since $xy-yx \in J$, therefore $C(R) \subseteq J$, and hence

$$C(R) \subseteq J \subseteq N.$$

Thus, The commutator ideal C(R) is nil and N is an ideal of R, by Lemma 2.

Using (1) we have

$$x \in R \text{ implies that } x \in N \text{ or } x= e_1 e_2 \dots e_n , \quad e_i \text{ is idempotent.}$$

Moreover, $x \notin N$ implies that

$$x+N= e_1. e_2 \dots e_n + N$$
$$= (e_1 +N).(e_2+N)\dots(e_n+N)$$

and clearly, $(e_i+N)^2=e_i+N$.

So every element x +N in R/N is multiplicatively generated by idempotents, and since R/N is finite, R/N is Boolean, by Theorem B. This completes the proof of Theorem 1.

Theorem 2. Let R be a finite ring such that

(2) The set R\ J is multiplicatively generated by idempotents.

Then N is an ideal of R and R/N is Boolean.

The following Lemmas are needed for the proof of Theorem 2.

Lemma 4. Any homomorphic image of a ring satisfying (2) must satisfy (2).

Proof. Let $\phi : R \rightarrow R'$ be a surjective ring homomorphism. Let $x' \in R' = \phi(R)$, then $x' = \phi(x)$ for some $x \in R$. Now using (2),

$$x \in J \; or \; x = e_1.e_2...e_n \text{ , where } e_i \text{ is idempotent in R.}$$

Let J' denote the Jacobson radical of R'. It is readily verified that $\phi(J) \subseteq J'$, and thus, for all $x' = \phi(x) \in R' = \phi(R)$,

$$x' = \phi(x) \in \phi(J) \subseteq J' \; or \; \phi(x) = \phi(e_1 e_2 ... e_n).$$

Thus for all $x' = \phi(x) \in R' = \phi(R)$

$x' \in J'$ or x' is a product of idempotents.

So, R' satisfies (2).

Lemma 5. Let R be a ring with identity such that R satisfies (2).

Then R is Boolean.

Proof. Let a $\in$ J. Then a+1$\notin$ J, and hence

$$a + 1 = e_1.e_2...e_n \text{ , } e_i \text{ is idempotent.}$$

So, $e_1(a+1) = e_1.e_2...e_n = a+1$, and thus e_1=1, since a+1 is invertible.

Continue this way to show that $e_2=e_3=\ldots=e_n=1$, which implies that a=0.

So, J=0, and thus (2) implies that R is multiplicatively generated by idempotents.

Therefore, R is Boolean, by Theorem A.

Proof of Theorem 2

Let R be a finite ring satisfying (2).

Case 1. If J=0, then R is a semisimple Artinian ring and hence has an identity. Thus, by Lemma 5, R is Boolean and the Theorem follows.

Case 2. Suppose J ≠ 0. Then R/J is a semisimple Artinian ring and satisfies (2) by Lemma 4. Hence R/J is a ring with identity satisfying (2), and thus R/J is Boolean by Lemma 5.

So, R/J is commutative, and thus

$xy - yx \in J$, for all x, y in R.

Hence $C(R) \subseteq J$. But J is nilpotent, since R is Artinian. So

$C(R) \subseteq J \subseteq N$,

and hence the commutator ideal C(R) is nil, which implies that N is an ideal of R, by Lemma 2.

To complete the proof, since $J \subseteq N$, $x \notin N$ implies that $x \notin J$ and by (2)

$x = e_1 . e_2 \ldots e_n$, e_i is idempotent.

Hence $x \notin N$ implies that

$$x + N = e_1 . e_2 \ldots e_n + N = (e_1 + N).(e_2 + N)\ldots(e_n + N)$$

where, $(e_i + N)^2 = e_i + N$.

Thus, R/N is multiplicatively generated by idempotents, and since R/N is finite, R/N is Boolean, by Theorem B.

Theorem 3. Let R be a ring with identity 1 such that

(3) The set R\C is multiplicatively generated by idempotents.

Then R is commutative.

Proof. Let e be any idempotent of R and x any element of R. Then,

$$a = ex - exe \in N.$$

Suppose that $a \notin C$, then $1 + a \notin C$, and hence using (3),

$1 + a = e_1 . e_2 \ldots e_n$, where e_i is idempotent. So $e_1(1 + a) = e_1 . e_2 \ldots e_n = 1 + a$. This implies that $e_1 = 1$ since $1 + a$ is a unit. Similarly, we get $e_2 = e_3 = \ldots = e_n = 1$. So $a = 0$ *and* $ex = exe$. Similarly we can show that $xe = exe$, *and hence* $e \in C$.

On the other hand, if $a \in C$, then it is easy to show that also the idempotents are central.

So, in all cases, the idempotents are central which implies using (3) that R is commutative.

Theorem 4: Let R be a ring such that

(4) The set R\ J is multiplicatively generated by idempotents and nilpotents.

If R satisfies the polynomial identity $x^m = x^{m+1} f(x)$, where f is a polynomial with integer coefficients, and if the set N of nilpotent elements is commutative, then N is an ideal of R and R/N is Boolean.

Proof.

A simple induction shows that $x^m = x^{2m} f(x)^m$. This implies that $e = x^m f(x)^m$ is idempotent for each x in R.

Let $x \in J$. Since J is an ideal then $e = x^m f(x)^m$ is an idempotent element in J. But J contains no nonzero idempotent elements and hence e = 0.

So, $x^m f(x)^m = 0$, and hence $x^m = x^{2m} f(x)^m = 0$.

So every element in J is nilpotent. This implies, using (4), that every element in R is multiplicatively generated by idempotents and nilpotents.

Now, using Theorem C, N is an ideal and R/N is Boolean.

Corollary: Let R be a finite ring such that

(5) The set R\ (NUJ) is multiplicatively generated by idempotents.

Then N is an ideal of R and R/N is Boolean.

Proof.

Since R is Artinian (being finite), then the Jacobson radical J is nil and hence NUJ=N. This implies by (5) that R\N is multiplicatively generated by idempotents. Now, using Theorem 1, N is an ideal of R and R/N is Boolean.

The following example shows the essentiality that R is finite in Theorems 1 and 2, and the essentiality that R has identity in Lemmas 3 and 5.

<u>*Example*</u> Let R_o be a ring with identity, and let R be the ring of all ∞ x ∞ matrices over R_o in which at most a finite number of entries are non zero. For every X $\in$ R, there exists a finite n x n matrix A over R_o such that

X = $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ where 0's are zero matrices.

Let L , M and N be matrices in R defined by

L = $\begin{pmatrix} I_n & I_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, M = $\begin{pmatrix} 0_n & 0_n & 0 \\ A & I_n & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and N = $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$

It is readily verified that L, M, and N are idempotents.

Clearly for every X in R, X = LMN. Thus, every X in R can be expressed as a product of idempotents. However, N is not an ideal, since

$$S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R, \text{ and } T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N, \text{ but } ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not in N.}$$

Please note that related and useful results can be found in references [3-14].

3 Conclusion

In conclusion, we would like to express our gratitude to the referees for their valuable suggestions.

Competing Interests

Author has declared that no competing interests exist.

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