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Problem Solving Framework for Mathematics Discipline

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

This paper identifies a 4-step framework that can be implemented in almost every mathematics lesson and training setting to move learners towards problem solving effectively. This framework which is built upon existing ideas proposed over the years in the mathematics education discipline and best practices concerning cognitive development and effective teaching and learning environment including solved examples provides teachers with very useful guidelines for classroom instruction. Ultimately, this framework can be used to move students towards an active learning environment which is more effective and enjoyable for teachers and students for learning.

Keywords: Problem – solving approach; mathematics education; framework; mathematical problems; George Polya; mathematical methods.

1 Introduction

Getting involved in a task for which there is no immediate answer is termed as problem solving [1]. Problem solving is, simply stated, the ability to analyse and find a solution to a problem at hand. [2] describe problem

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solving as the process by which students explore non-routine questions. The explorations involve using a wide range of strategies to solve unfamiliar tasks, as well as developing the processes of analysing, reasoning, generalizing and abstracting. In the exploration process students make mistakes and backtrack. [3] also shares the view that problem solving is a principal instructional strategy used to fully engage students in important mathematics learning situation. It also goes beyond the domain of mathematics to include everyday life activities in general. Mathematical problem solving can also be described as finding a way around a difficulty, around an obstacle, and finding a solution to a problem that is unknown [4]. [5] suggested that reasoning and/or higher order thinking must occur during mathematical problem solving. In addition, [6] are of the view that, for tasks to be considered as mathematical problem solving, they must be developmentally appropriate for students. A challenging problem solving task for a grade 2 students may only be a routine word problem for a grade 3 student. [7] also indicated that the mode of instructions that are suitable for placing more emphasis on problem solving strategies, application and higher order thinking skills is what students need. That is students should be encouraged to participate in their mathematics classrooms in order to build self-confidence and to use mathematics appropriately in their daily lives [8]. For instance, in mathematics classrooms, allowing students to work in groups in problem solving situations help them to analyse situations, to pose questions, check for accurate results and try different strategies. Polya's [9] "How to Solve It" book suggested four areas of problem-solving, which have become a widely used framework often recommended for teaching and assessing problem-solving skills by some researchers. These are:

i. Understanding the problem,
ii. Devising a plan to solve the problem,
iii. Implementing the plan, and
iv. Reflecting on the problem.

It is based on the ideas of [9,10,11] and the Institute for Advanced Study/Park City Mathematics Institute International Seminar [12] that this paper identifies a 4-step framework that can be implemented in all mathematics lessons towards problem solving environment. This framework is built upon existing ideas and best practices in cognitive development and effective learning environments including solved examples in each step of the framework to provide mathematics teachers with very useful guidelines to move students from lecture-based learning environment towards an active problem solving environment.

2 Techniques that Encourage Problem Solving

In the view of [13], most of the talking, the questioning and the thinking are done by the teacher in the classroom, placing the students in a passive role rather than an active role. The lecture format of learning is a popular and common approach to delivery of content in schools. But this form of learning does not encourage critical thinking and problem solving on the part of students. However, [14] posited that with the lecture format of learning, increasing students' critical thinking is very difficult. Discussions of topics are done sequentially rather than critically analysing them. The lecture method encourages the delivery of large amounts of information at a time and the students tend to memorize the material. Among new teachers, lecture method is common because it comes with a strong academic tradition and its teacher centred as well [14].

Teachers must give up the notion that students cannot learn the topics at hand unless the teacher covers it and adopt a proper way of teaching that encourages active learning and better understanding. Adopting problem solving as an instructional strategy brings about an active learning environment that makes learning more enjoyable for both students and teachers and also promotes critical thinking among students. In [8], they declared that the role of teachers has been transformed to guide and facilitate teaching and learning in the classroom rather than serving as a source of knowledge and authority. [15] also outline some benefits of teaching mathematics through problem solving which include:

I. Helping students to understand that mathematics develops through sense-making process;
II. Deepening students' understanding of underlying mathematical ideas and methods and;
III. Engaging students' interest (p. 20)

Carpenter [16] is of the view that, the aims of teaching through problem solving are to encourage students to refine and build onto their own processes over a period of time as their experiences allow them to discard some ideas and become aware of further possibilities. However, learning through problem-solving will enable students to deal more effectively and successfully with most types of mathematical problems. Problem solving instructional practices force students to become active participants in the learning process and engage teachers to participate actively as learners in the classroom along with students.

There have been many problem solving techniques proposed over the years. [10] designed, implemented and evaluated a learning environment that emphasized students' acquisition of an overall metacognitive strategy for solving mathematical problems involving five stages. These are:

i. Build a mental representation of the problem
ii. Decide how to solve the problem
iii. Execute the necessary calculations
iv. Interpret the outcome and formulate an answer
v. Evaluate the solution

Taking these stages into account during instruction is helpful and can improve students' problem solving ability. [17] declared that during the problem solving process, students consciously interpret a problem, analyse the important attributes and decide on a solution. [18] explains that problem solving skills entails more than drawing on one background knowledge, instead information must be effectively applied to new problem situations. According to [11] the success in solving a problem is directly related to the choice of the appropriate strategy. The most frequent solution strategies are:

i. Making systematic list
ii. Drawing diagram
iii. Looking for pattern
iv. Making a table
v. Writing equation
vi. Reasoning and simplifying the problem and
vii. Guess and check ([11])

In addition, The Institute for Advanced Study/Park City Mathematics Institute International Seminar (IAS/PCMIIS) [12] also outlined the process of solving mathematical problems to include:

I. Understanding and defining the problem,
II. Exploring the problem,
III. Hypothesizing,
IV. Testing and formalizing,
V. Reflecting/evaluating

Teachers should provide many opportunities for students to engage in problem solving where critical thinking takes place to provide the greatest benefit to students. Most mathematics teachers are with the belief that developing problem solving techniques in their students is of primary importance; however few have an idea exactly what it is, how it should be taught and how it should be assessed. The following framework (Fig. 1) is a 4-step guideline with examples that can be implemented in any mathematics classroom or training setting to help students acquire problem-solving skills.

3 The 4-step Model to Move Students towards Problem Solving

Modern day teacher need not to cover topics for students; however the teachers should be seen discovering topics and ideas with students in an active learning environment to promote problem solving and critical thinking among learners. According to [9], teachers should make sure students are able to answer questions like; Do you understand all the words used in stating the problem?, What are you asked to find or show?, Can you restate the problem in your own words?, Can you think of a picture or a diagram that might help you understand the problem better?, Is there enough information to help find a solution to the problem?

3.1 Step 1: Holistic understanding of the problem

The main objective of this step is to guide students and pupils to gain better understanding of any mathematical problem at hand. This involves the identifying of clues or key words in mathematical problems. Understanding the problem requires not only knowing what to do but also the key pieces of information that somehow need to be put together to obtain the solution. [19] declared that there is no chance of being able to solve a problem unless you understand the problem first. For better understanding of a mathematical problem, the learner should be able to answer the following questions: What are the known and the unknown? And what are the conditions? Before the start of a problem solving class, it is advisable that students are put in groups to make sure they all take part in the discussion.

Here are some of the few examples that throw more light on the holistic understanding of the problem.

Question 1: The sum of the length of the edges of a cube is 48 cm. Find the length of each edge.

Analysis; Teacher ask the students to read the statement carefully and come out with their clues or key words in the statement. Majority of the students came out that the key words as ‘sum’ and ‘edges of a cube’. The teacher then asks the students again that, how many edges does a cube has? And the students also responded that, a cube has four equal edges with some of them referring to a cube of sugar as an example. The students also concluded that ‘sum’ means addition. The students groups together came out with their understanding as; when all the length of the sides of a cube is put together you will have 48 cm.

Question 2: The weight of an object varies inversely as the square of its distance from the centre of the earth.

Analysis; The teacher gives students time to read the question carefully and come out with their clues or key words and their understanding about the question. The students came out with the Key word; varies inversely, meaning the weight of an object increases (or decreases) as the square of its distance decreases (or increases) from the centre of the earth. The students groups together came out with their understanding as; the weight of an object depends on its distance from the centre of the earth.

3.2 Step 2: Identify method(s) for solution

The main objective of this step is to guide students on identifying a method(s) or a strategy for finding solutions to mathematical problems base on their understanding.This method(s) or strategy includes drawing sketches/diagrams, formulas, looking for patterns, and so on. The teacher guide the students on making of sketches or drawing diagrams about the given problems where necessary, looking for or identifying appropriate methods or formulas where they are required to help find solutions to the given mathematical problems and looking for patterns where necessary in finding solutions to problems. The teacher should make students aware that, always they should look out for methods or strategies that are useful to the problems at hand in order for them to save time and look out for better solution options.

Below are some of the few examples that one can go through for better understanding of this second step.

Question 1: Yaw is now five times as old as Kwame. In ten years’ time, Yaw will be three times as old as Kwame. How old are they?

Analysis; The teacher guides the students to read the question carefully and come out with methods or strategies for solving the problem at hand base on their understanding. With the guidance from the teacher, the students came out with these method or strategy below;

Let Kwame's current age be n and Yaws current age = $5n$.

The teacher ask the students to come up with their ages in ten years' time, the students groups agreed that, in ten years' time, the age of Kwame will be $(10 + n)$ and that of Yaw will be$(10 + 5n)$.

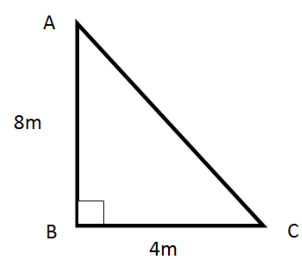
Again, the teacher asks the students to bring out a general method or strategy for solving the problem depending on their understanding of the problem. Upon many deliberations among the groups of the students, the students concluded that, the appropriate method or strategy for finding solution to the problem is

$$10 + 5n = 3(10 + n).$$

The teacher should make the students aware that, the age in the question above can be represented by any variable apart from the variable n.

Question 2: A telephone pole is supported by a wire attached to the pole 8m above the ground and to a point on the ground 4m from the foot of the pole. Calculate the total length of the wire.

Analysis; The teacher guides the students to read the question carefully and come out with methods or strategies for solving the problem base on their understanding. The students groups together agreed to sketch a diagram for the problem for better understanding. With the guidance of the teacher, the students came out with the diagram below;



After a careful observation of the diagram, the students again came out with the following ideas;

Let the side $|AB| = 8m$ represent the length of the telephone pole, $|BC| = 4m$ representing the distance between the wire and the foot of the telephone pole on the ground and $|AC| =?$ representing the total length of wire supporting the telephone pole from the top to the ground. The teacher further explains Pythagoras theorem principle to the students which states that in a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. With the guidance of the teacher again, the students were able identify the side $|AB|$ as opposite, side $|BC|$ as adjacent and side $|AC|$ as the hypotenuse.

The teacher encourages the students to come up with a method or a strategy to solve the problem base on the idea of Pythagoras theorem principle. Upon further discussion among the students themselves and a careful study of the diagram, the students came out with the appropriate method for finding solution to the problem as:

$$|AC|^2 = |AB|^2 + |BC|^2$$

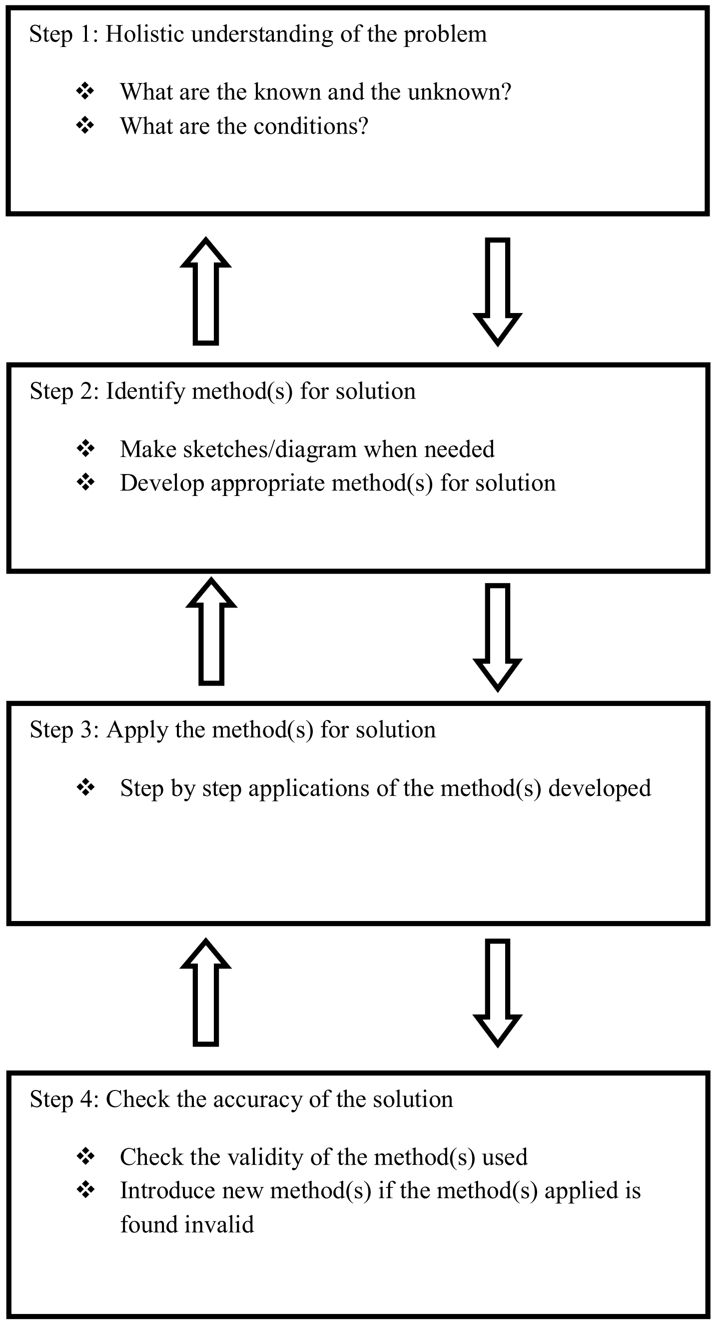


Figure 1. The 4-step model structure to direct learners towards problem solving

3.3 Step 3: Apply the method(s) for solution

The main objective of this third step was to guide students on how to use various methods or strategies in finding solutions to mathematical problems.The teacher must take the students through how to use various methods or strategies in finding solutions to the problems given based on the information available in the questions. This step purposely teach students how to solve mathematical problems step by step through the application of appropriate problem solving methods like sketches or diagrams, formulas and patterns.

Here are some of the examples that throw more light on the application of appropriate methods in solving mathematical problems.

Question 1: The sum of four consecutive odd numbers is 16. Find the numbers.

Analysis; The students were given ample time by the teacher to read the question carefully and come up with a method or a strategy to solve the problem base on their experience in step 2. This was their response;

Let w be the first odd number, and the other numbers were given by the students as;

$$(w+2),(w+4),(w+6)$$

With the guidance of teacher the students again came out with the method for finding solution to the question base on the key word 'sum' as:

$$w+(w+2)+(w+4)+(w+6)=16.$$

The teacher asks the students to discuss how to solve the question based on the method given above among themselves in their various groups. With the guidance of the teacher, the students together came out with the following solution steps below;

Expand the bracket to get;

$$w+w+2+w+4+w+6=16.$$

Group like terms to get;

$$w+w+w+w=16-6-4-2,$$

$$4w=4\ .$$

Divide both sides of the equation $4w=4$ by 4 to get;

$$w=1$$

The students gave the four consecutive odd numbers as:

$$1,3,5\ and\ 7.$$

Question 2: The 5th and 13th terms of an Arithmetic Progression are -2 and 14 respectively. Find the Arithmetic Progression.

Analysis; The teacher takes the students through the topic Series and Sequence which involves Arithmetic Progression and Geometric Progression. The students were given enough opportunity to ask any question bothering their minds about the topic. The students were given enough time by the teacher to read the question carefully and come up with a method or a strategy to solve the problem base on their experience in step 2. Upon further discussion among the members of each group, the students agreed on two methods for the two different terms.

For 5th term the method is

$U_5 = a + (5-1)d = -2$

$a + 4d = -2 \quad \rightarrow \quad (1)$

For 13th term the method is

$U_{13} = a + (13-1)d = 14$

$a + 12d = 14 \quad \rightarrow \quad (2)$

The teacher guides the students on bringing out the solution steps of this problem as indicated below;

The students groups concluded that the two equations should be solved by using elimination method in the simultaneous equation principle. They further suggested that, the equations (1) and (2) should be multiplied by -3 and 1 respectively to get:

$-3a - 12d = 6 \quad \rightarrow \quad (3)$

$a + 12d = 14 \quad \rightarrow \quad (2)$

The students again suggested the addition of equations (3) and (2) to get:

$-3a + a - 12d + 12d = 6 + 14$

$-2a = 20$

Divide both sides of the equation by -2 to get:

$a = -10$

Substitute $a = -10$ into equation (1) to get:

$-10 + 4d = -2$

Group like terms to get:

$4d = -2 + 10$

$4d = 8$

Divide both sides of the equation by 4 to get:

$d = 2$

Upon finding solutions to the first term (*a*) and the common difference (*d*), the students agreed that, the Arithmetic Progression is;

$-10, -8, -6, -4, -2$ ...

3.4 Step 4: Checking the accuracy of the solution

The objective of this step is to guide students on how to check whether the solutions found appropriately or accurately answers the mathematical problems given or not. The teacher guides the students through how to check whether the solutions found accurately answers the problem given or not based on the information available in the questions. This step will enable students adapt to a proper identification of methods or strategies since it gives you the opportunity to check its accuracy at all times. In situations where the methods or strategies are identified to be inaccurate or inappropriate for that question, different methods or strategies are introduced until one of them is found valid for providing accurate solution to the actual question at hand. This step purposely teaches students how to check the accuracy or the validity of appropriate problem solving methods or strategies like sketches or diagrams, formulas and patterns on finding solution to mathematical problems.

Here are some examples for further elaborations of this step. These examples were derived from the solutions in the step 3 examples used above.

Question 1: the sum of four consecutive odd numbers is 16. Find the numbers.

Analysis; The students came up with the solutions for the four consecutive odd numbers as 1, 3, 5 and 7 in the previous step.

The teacher asks the students to substitute the value of the first odd number into the method or add all the four odd numbers to check the accuracy of the method adopted for the solution of the problem. The students together came up with the solution below:

$$w + (w + 2) + (w + 4) + (w + 6) = 16$$

$$1 + 3 + 5 + 7 = 16$$

$$16 = 16$$

The students concluded that the method used is appropriate for giving accurate solution which showed that the right hand side of the equal sign is the same as the left hand side.

Question 2: The 5th and 13th terms of an Arithmetic Progression are -2 and 14 respectively. Find the Arithmetic Progression.

Analysis; from the step 3 the students groups together came up with the values for the first term (*a*) and the common difference (*d*) of the question above as *-10* and *2* respectively.

The teacher asks the students to substitute the values of the first term (*a*) and the common difference (*d*) into the methods to check the accuracy of the methods adopted for the solution of the problem. The students together came up with the solution below:

For the 5th term,

$$U_n = a + (n - 1)d$$

$$U_5 = -10 + (5 - 1)2$$

$$U_5 = -10 + 8$$

$$U_5 = -2$$

For the 13th term,

$$U_{13} = -10 + (13 - 1)2$$

$$U_{13} = -10 + 24$$

$$U_{13} = 14$$

The students concluded that the methods used are appropriate for giving accurate solution. Following the same procedure, it can be concluded that, the sequence $-10, -8, -6, -4, -2$ ... is an accurate solution.

4 Discussions and Conclusion

When teachers think about what should happen in a classroom, it is important to consider the kinds of active learning that can encourage problem solving. It will be in the good interest of learners that teachers give a careful consideration to current instructional strategies and to the personal beliefs that move them prior to contemplating this particular approach to teaching. Implementing problem solving through this framework plainly requires a strong commitment to a student centred learning which may be uncomfortable and unfamiliar to both teachers and learners. Other hindrances in the application of this framework can be overcome through creativity and proper planning. Although time constraints and class size may limit the frequency and duration of techniques that encourage problem solving; however, it is still very possible to engage learners in groups.

Moreover, problem-solving can be accomplished in any mathematics classroom by modification of teaching and the incorporation of active learning techniques. Using the 4-step framework to help learners acquire problem solving skills will bring about a fundamental change in instructional techniques from that of the traditional teacher-centred method to a more student-centred method which will result in learning experience which is both more valuable and enjoyable.

Competing Interests

Authors have declared that no competing interests exist.

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