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Convergence of the Ishikawa Type Iteration Process with Errors of I-Asymptotically Quasi-nonexpansive Mappings in Cone Metric Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. Authors AUR and MU designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors AUR, KQ and GM managed the analyses of the study. Author GM managed the literature searches. All authors read and approved the final manuscript.

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Abstract

The goal of this article is to consider an Ishikawa type iteration process with errors to approximate the fixed point of I-asymptotically quasi non-expansive mapping in convex cone metric spaces. Our results extend and generalize many known results from complete generalized convex metric spaces to cone metric spaces.

Keywords: Ishikawa type iteration; I-asymptotically quasi-nonexpansive mapping; asymptotically nonexpansive mapping; cone metric space; normal and nonnormal cone; fixed point.

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1 Introduction and Preliminaries

Recently, Huang and Zhang [1] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. The category of cone metric spaces is larger than metric spaces and there are different types of cones. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. Subsequently, various authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal non-normal cones. There exists a lot of works involving fixed points used the Banach contraction principle.

Consistent with [2] and [1], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

1. P is closed, nonempty and $P \neq \{\theta\}$;
2. $\gamma, \mu \in \mathbb{R}, \gamma, \mu \geq 0$ and $u, v \in P$ imply $\gamma u + \mu v \in P$;
3. $u \in P$ and $-u \in P \Longrightarrow u = 0 \Leftrightarrow P \bigcap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preccurlyeq y$, while $x \ll y$ will stand for $y - x \in P^0$ (interior of P). If $P^0 \neq \phi$ then P is called a solid cone (see, [3]).

There exists two kinds of cones- normal (with normal constant K) and non-normal cones [2].

Let E be a real Banach space, $P \subset E$ a cone and $\preccurlyeq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \preccurlyeq x \preccurlyeq y \text{ imply } \|x\| \leq K\|y\| \tag{1.1}$$

or equivalently, if, for all $n, x_n \preccurlyeq y_n \preccurlyeq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x \tag{1.2}$$

The least positive number K satisfying (1.1) is called the normal constant of P. It is clear that $K \geq 1$.

Example 1.1. (see [3]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $= \{x \in E: x(t) \geq 0 \geq\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preccurlyeq x_n \preccurlyeq y_n$ and $\lim_{n\to\infty} y_n = \theta$, but

$$\|x_n\| = \max_{t\in[0,1]} \left|\frac{t^n}{n}\right| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1,$$

hence x_n does not converge to zero. It follows by (1.2) that P is a normal cone.

Definition 1.1. (see [1] and [4]) Let X be a nonempty set. Suppose that the mapping $d: X \times X \to E$ satisfies:

(1). $\theta \leq d(x, y), \forall\, x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;
(2). $d(x, y) = d(y, x), \forall\, x, y \in X$;
(3). $d(x, y) \preccurlyeq d(x, z) + d(z, y), \forall\, x, y, z \in X$.

Then d is called a cone metric [1] or K-metric [4] on X and (X, d) is called a cone metric space [1] or K-metric [4] (we shall use the first term).

The concept of a cone metric space is more general than that of a metric space because each metric space is a cone metric space, where $E = \mathbb{R}$ and $P[0,+\infty)$.

Example 1.2.

1. Let $E = \mathbb{R}^2, P = \{(x,y) \in E: x,y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d: X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is cone metric space [1] with normal cone P where $K = 1$ (see [5]).
2. For other examples of cone metric spaces, one can see [4], pp. 853-854.

Definition 1.2. (see [1]) Let (X,d) be a cone metric space. We say that $\{x_n\}$ is

1. A Cauchy sequence, if for every $c \in E$ with $\theta \ll c$, $\exists$ an N such that for all $m,n > N$, $d(x_n, x_m) \ll c$.
2. A convergent sequence, if for every $c \in E$ with $\theta \ll c$, $\exists$ an N such that for all $n > N, d(x_n, x) \ll c$ for fixed $x \in X$.
3. A cone metric space X is said to be complete, if every Cauchy sequence in X is convergent in X.

Let us recall [1] that P is a normal solid cone, then $x_n \in X$ is a Cauchy sequence if and only if $\|d(x_n, x_m)\| \to 0$ as $n, m \to \infty$. Further, $x_n \in X$ converges to $x \in X$ if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

In the sequel, we assume that E is a real Banach space and that P is a normal solid cone in E, that is, normal cone with $P^0 \neq \phi$. The last assumption is necessary in order to obtain reasonable results connected with convergence and continuity. The partial ordering induced by the cone P will be denoted by $\preccurlyeq$.

2 Convexity in Cone Metric Space

Let (X,d) be a cone metric space with a solid cone P and C be a nonempty closed convex subset of X. A mapping $T: C \to C$ is called asymptotically nonexpansive if there exists $k_n \in [0,\infty), \lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, T^n y) \preccurlyeq (1 + k_n) d(x,y), \forall x, y \in C.$$

Let $F(T) = \{x \in X: Tx = x\}$, if $F(T) \neq \phi$, then T is called asymptotically quasi-nonexpansive, if $\exists\, k_n \in [0,\infty), \lim_{n\to\infty} k_n = 0$ such that $d(T^n x, p) \preccurlyeq (1 + k_n) d(x,p), \forall\, x \in C$ and $p \in F(T)$.

Furthermore, it is *I*-asymptotically quasi-nonexpansive, if there exists sequence $\{v_n\} \subset (0,\infty)$ with $\lim_{n\to\infty} v_n = 0$ such that

$$d(T^n x, p) \preccurlyeq (1 + v_n) d(I^n x, p), \forall\, x \in C, p \in F(T)$$

where $I: C \to C$ be asymptotically nonexpansive mappings with $\{v_n\} \subset (0,\infty)$.

All of the above mappings are contractive mappings. From the above definitions it is easy to see that if $F(T)$ is non-empty, an asymptotically nonexpansive mappings must be asymptotically quasi-nonexpansive mappings, but the converse does not hold [6].

In recent years, asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied extensively in the setting of convex metric spaces [7,8,9,5] and [10].

In 1970, Takahashi [11] first introduced the notion of convex metric space which is more general space. It should be pointed out that each linear normed space is a special example of convex metric space, but there exists some convex metric spaces which cannot be embedded into normed spaces [11].

Now we introduced the following.

Definition 2.1. Let (X,d) be a cone metric space, and $I=[0,1]$. A mapping $W: X^2 \times I \to X$ is said to be convex structure on X, if for any $(x,y,\lambda) \in X^2 \times I$ and $u \in X$, the following inequality holds:

$$d(W(x,y,\lambda),u) \preccurlyeq \lambda d(x,u) + (1-\lambda)d(y,u)$$

If (X,d) is a cone metric space with a convex structure W, then (X,d) is called a convex abstract metric space or convex cone metric space. Moreover, a nonempty subset C of X is said to be convex if $W(x,y,\lambda) \in C$, for all $(x,y,\lambda) \in C^2 \times I$.

Definition 2.2. Let (X,d) be a cone metric space, $I=[0,1]$, and $\{a_n\},\{b_n\},\{c_n\}$ be real sequences in $[0,1]$ with $a_n+b_n+c_n=1$. A mapping $W: X^3 \times I^3 \to X$ is said to be convex structure on X, if for any $(x,y,z,a_n,b_n,c_n) \in X^3 \times I^3$ and $u \in X$, the following holds:

$$d(W(x,y,z,a_n,b_n,c_n),u) \preccurlyeq a_n d(x,u) + b_n d(y,u) + c_n d(z,u)$$

If (X,d) is a cone metric space with a convex structure W, then (X,d) is called a generalized convex cone metric space. Moreover, a nonempty subset C of X is said to be convex, if $W(x,y,z,a_n,b_n,c_n) \in C$, for all $(x,y,z,a_n,b_n,c_n) \in X^3 \times I^3$.

Remark 2.1. If $E=\mathbb{R}, P=[0,+\infty), \|\,.\,\| = |\,.\,|$, then (X,d) is a convex metric space, i.e., generalized convex metric space.

Example 2.1. Let (X,d) be a cone metric space as in Example 1.2(a). If $(x,y,\lambda)=\lambda x+(1-\lambda)y$, then (X,d) is a convex cone metric space. Hence, this concept is more general than that of a convex metric space.

Definition 2.3. Suppose that (X,d) be a cone metric space with a convex structure $W: X^3 \times I^3 \to X$ and $f: X \to X$ be I-asymptotically quasi-nonexpansive mapping, $I: X \to X$ be asymptotically nonexpansive mapping. Then an iteration scheme is the sequence of mappings $\{x_n\}$ defined by

$$x_{n+1} = W(x_n, I^n y_n, u_n, a_n, b_n, c_n)$$

$$y_n = W(x_n, f^n y_n, v_n, a_n', b_n', c_n') \tag{2.1}$$

where $\{a_n\},\{b_n\},\{c_n\}$, $\{a_n'\},\{b_n'\},\{c_n'\}$ are real sequences in $(0,1)$ with $a_n+b_n+c_n = a_n'+b_n'+c_n' = 1$ and $\{u_n\},\{v_n\}$ be two sequences in X satisfying the following condition: for any non-negative integers $n,m, 0 \le n \le m$; if $\delta(A_m) > 0$, then

$$\max_{n\ge i, j \ge m}\left\{\|d(x,y)\| : x \in \{u_i, v_i\}, y \in \left\{x_j, y_j, Iy_j, fx_j, u_j, v_j\right\}\right\} < \delta(A_m) \tag{2.2}$$

where $A_{nm} = \{x_i, y_i, Iy_i, fx_i, u_i, v_i : n \ge i \ge m\}$ and $\delta(A_{nm}) = \sup_{x,y \in A_{nm}} \|d(x,y)\|$.

Then $\{x_n\}$ is called Ishikawa type iteration scheme with errors.

In the sequel, we shall need the following lemma.

Lemma 2.1. (see [12]) Let $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\mu_n\}$ be four nonnegative real sequences satisfying

$$\alpha_n \preccurlyeq (1+\gamma_n)(1+\mu_n)\alpha_n + \beta_n, \forall\, n \ge 1$$

If $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$, then $\lim_{n\to\infty}\alpha_n$ exists.

3 Main Results

Before proving our main results of this paper, we shall prove the following lemma.

Lemma 3.1. Let X be a convex cone metric space, C be a nonempty closed convex subset of X, $f: C \to C$ be I-asymptotically quasi-nonexpansive mapping with sequence $\{k_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $I: C \to C$ be asymptotically nonexpansive mapping with $\{k_n^{'}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} k_n^{'} < \infty$ and $F = F(f) \cap F(I)$. The Ishikawa type iteration sequence $\{x_n\}$ is generated by (2.1), $\{u_n\}, \{v_n\}$ satisfying (2.2). If $F \neq \phi$ and $\sum_{n=1}^{\infty}(c_n + c_n^{'}) < \infty$, then

(i) there exists a constant vector $v \in P \backslash \{\theta\}$ such that

$$\|d(x_{n+1}, p)\| \preccurlyeq k(1+\alpha_n)(1+\beta_n)\|d(x_n, p)\| + k\|v\|\gamma_n$$

for all $n \in \mathbb{N}$ and for all $p \in F$, where k is the normal constant of a cone P.

(ii) there exists a real number $M > 0$ such that

$$\|d(x_m, p)\| \preccurlyeq kM\|d(x_n, p)\| + kM\|v\| \sum_{i=n}^{m-1} \gamma_j$$

for all $n, m \in \mathbb{N}$ and for all $p \in F$, where k is the normal constant of a cone P.

Proof. First, we will show that F is closed. Let $\{\xi_n\} \subset F$ be such that $\xi_n \to x$ as $n \to \infty$.

In addition,

$$\begin{aligned}\|d(fx, x)\| &= k\|d(fx, f\xi_n)\| + k\|d(\xi_n, x)\| \\ &\preccurlyeq (1 + u_1 v_1)k\|d(\xi_n, x)\|\end{aligned}$$

This implies that $fx = x$. By the same way, we can get $x \in F(I)$. So, x is a common fixed point of f and I. Thus F is closed set.

(i) Now, for any $p \in F = F(f) \cap F(I) \neq \phi$,

$$\begin{aligned}d(y_n, p) &= d(W(x_n, f^n x_n, v_n; a_n^{'}, b_n^{'}, c_n^{'}), p) \\ &\preccurlyeq a_n^{'} d(x_n, p) + b_n^{'} d(f^n x_n, p) + c_n^{'} d(v_n, p) \\ &\preccurlyeq (1 - b_n^{'}) d(x_n, p) - c_n^{'} d(x_n, p) + c_n^{'} d(v_n, p) + b_n^{'} d(f^n x_n, p) \\ &\preccurlyeq (1 - b_n^{'}) d(x_n, p) + c_n^{'} d(x_n, v_n) + b_n^{'}(1 + k_n)(1 + k_n^{'}) d(x_n, p) \\ &\preccurlyeq \left(1 + b_n^{'}(k_n + k_n^{'} + k_n k_n^{'})\right) d(x_n, p) + c_n^{'} d(x_n, v_n) \end{aligned} \tag{3.1}$$

$$\begin{aligned}d(x_{n+1}, p) &= d(W(y_n, I^n y_n, u_n; a_n, b_n, c_n), p) \\ &\preccurlyeq a_n d(y_n, p) + b_n d(I^n y_n, p) + c_n d(u_n, p) \\ &\preccurlyeq (1 - b_n) d(y_n, p) - c_n d(y_n, p) + c_n d(u_n, p) + b_n d(I^n x_n, p) \\ &\preccurlyeq (1 - b_n) d(y_n, p) + c_n d(y_n, u_n) + b_n(1 + k_n^{'}) d(y_n, p) \\ &\preccurlyeq (1 + b_n k_n^{'}) d(y_n, p) + c_n d(y_n, u_n) \end{aligned} \tag{3.2}$$

Substituting (3.1) into (3.2), it can be obtained that

$$
\begin{aligned}
d(x_{n+1},p) &\preccurlyeq (1+\alpha_n)[(1+\beta_n)d(x_n,p)+c_n^{'}d(x_n,v_n)]+c_nd(y_n,u_n)\\
&\preccurlyeq (1+\alpha_n)(1+\beta_n)d(x_n,p)+(1+\alpha_n)c_n^{'}d(x_n,v_n)+c_nd(y_n,u_n)\\
&\preccurlyeq (1+\alpha_n)(1+\beta_n)d(x_n,p)+\gamma_n v
\end{aligned}
$$

where $\alpha_n = b_n k_n^{'},\ \beta_n = b_n^{'}(k_n + k_n^{'} + k_n k_n^{'}), \gamma_n = c_n + c_n^{'}$ and $v = (1+k_n)[d(x_n,v_n)+d(y_n,u_n)]$.

Now (i) follows from (1.1), where k is a normal constant of the cone P.

(ii) If $x \geq 0$, then $1 + x \leq e^x$. Therefore from (3.3),

$$
\begin{aligned}
d(x_m,p) &\preccurlyeq (1+\alpha_{m-1})(1+\beta_{m-1})d(x_{m-1},p)+\gamma_{m-1}v\\
&\preccurlyeq (1+\alpha_{m-1}+\beta_{m-1}+\alpha_{m-1}\beta_{m-1})d(x_{m-1},p)+\gamma_{m-1}v\\
&\preccurlyeq e^{\alpha_{m-1}+\beta_{m-1}+\alpha_{m-1}\beta_{m-1}}d(x_{m-1},p)+\gamma_{m-1}v\\
&\preccurlyeq e^{\alpha_{m-1}+\beta_{m-1}+\alpha_{m-1}\beta_{m-1}}\left[e^{\alpha_{m-2}+\beta_{m-2}+\alpha_{m-2}\beta_{m-2}}d(x_{m-2},p)+\gamma_{m-2}v\right]+\gamma_{m-1}v\\
&\preccurlyeq e^{(\alpha_{m-1}+\alpha_{m-2})+(\beta_{m-1}+\beta_{m-2})+(\alpha_{m-1}\beta_{m-1}+\alpha_{m-2}\beta_{m-2})}d(x_{m-2},p)\\
&\quad +e^{\alpha_{m-1}+\beta_{m-1}+\alpha_{m-1}\beta_{m-1}}\gamma_{m-2}v+\gamma_{m-1}v\\
&\preccurlyeq e^{(\alpha_{m-1}+\alpha_{m-2})+(\beta_{m-1}+\beta_{m-2})+(\alpha_{m-1}\beta_{m-1}+\alpha_{m-2}\beta_{m-2})}d(x_{m-2},p)\\
&\quad +e^{\alpha_{m-1}+\beta_{m-1}+\alpha_{m-1}\beta_{m-1}}[\gamma_{m-1}+\gamma_{m-2}]v\\
&\preccurlyeq e^{(\alpha_{m-1}+\alpha_{m-2})+(\beta_{m-1}+\beta_{m-2})+(\alpha_{m-1}\beta_{m-1}+\alpha_{m-2}\beta_{m-2})}d(x_{m-2},p)\\
&\quad +e^{\alpha_{m-1}+\beta_{m-1}+\alpha_{m-1}\beta_{m-1}}[\gamma_{m-1}+\gamma_{m-2}]v\\
&\preccurlyeq \ldots\ldots\ldots\ldots\\
&\preccurlyeq e^{\sum_{j=n}^{m-1}\alpha_j+\beta_j+\alpha_j\beta_j}d(x_{m-2},p)+e^{\sum_{j=n}^{m-1}\alpha_j+\beta_j+\alpha_j\beta_j}\left(\textstyle\sum_{j=n}^{m-1}\gamma_j\right)v\\
&\preccurlyeq Md(x_{m-2},p)+Mv\left(\textstyle\sum_{j=n}^{m-1}\gamma_j\right)
\end{aligned}
\tag{3.3}
$$

where $M = e^{\sum_{j=n}^{m-1}\alpha_j+\beta_j+\alpha_j\beta_j}$

Further (ii) follows from (1.1), because P is a normal cone with normal constant k.

This completes the proof of the lemma.

Theorem 3.1. Let X be a convex cone metric space, $C, f, I, \{x_n\}$ be same as in Lemma 3.1 and $F \neq \phi$. The Ishikawa type iteration sequence (2.1), $\{x_n\}$ converges strongly to a common fixed point in F if and only if $\lim\ \inf_{n\to\infty}\|d(x_n,F)\| = 0$, where $\|d(x_n,F)\| = \inf\{d(x_n,p): p \in F\}$.

Proof. Since $\|d(x_n,F)\| = \inf\{d(x_n,p): p \in F\} \preccurlyeq \inf\{d(x_n,p): \lim_{n\to\infty} x_n = q \in F\} = 0$, it follows that the necessity of the condition is obvious. Thus, we will only prove the sufficiency.

From Lemma 3.1(i), we have

$$\|d(x_n,F)\| \preccurlyeq k(1+\alpha_n)(1+\beta_n)d(x_n,p)+k\|v\|\gamma_n$$

Notice that $\sum_{n=1}^{\infty} k_n < \infty, \sum_{n=1}^{\infty} k_n^{'} < \infty, \sum_{n=1}^{\infty}(c_n + c_n^{'}) < \infty$ for all $n \in \mathbb{N}$ and $\{u_n\}, \{v_n\}$ satisfying (2.2), we have that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\gamma_n = 0$. Also by Lemma 2.1, $\lim_{n\to\infty}\|d(x_n,F)\|$ exists.

According to the hypothesis, $\lim\ \inf_{n\to\infty}\|d(x_n,F)\| = 0$, hence we have $\lim_{n\to\infty}\|d(x_n,F)\| = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$. By Lemma 3.1(ii), there exists a constant $M > 0$ such that

$$\|d(x_m,p)\| \preccurlyeq kM\|d(x_n,p)\| + kM\|v\| \textstyle\sum_{j=n}^{m-1}\gamma_j \tag{3.4}$$

for all $p \in F$ and $m > n$.

Since $\lim_{n\to\infty}\|d(x_n, F)\| = 0$ and $\sum_{n=1}^{\infty}(c_n + c_n^{'}) < \infty$, there exists a constant N_1 such that for all $n \geq N_1$

$$\|d(x_m, F)\| \prec \frac{\varepsilon}{3k^2M} \text{ and } \sum_{j=n}^{m-1}\gamma_j \prec \frac{\varepsilon}{6k^2\|v\|M}$$

We note that, there exists $p_1 \in F$ such that $\left\|d\left(x_{N_1}, p_1\right)\right\| \prec \frac{\varepsilon}{3k^2M}$

It follows from (3.4) that for all $m > n > N_1$,

$$\begin{aligned}\|d(x_m, x_n)\| &\preccurlyeq k\|d(x_m, p_1) + d(x_n, p_1)\| \\ &\preccurlyeq k\|d(x_m, p_1)\| + \|d(x_n, p_1)\| \\ &\preccurlyeq k^2M\left\|d\left(x_{N_1}, p_1\right)\right\| + k^2\|v\|M\sum_{j=N_1}^{m-1}\gamma_j \\ &\preccurlyeq k^2M\left\|d\left(x_{N_1}, p_1\right)\right\| + k^2\|v\|M\sum_{j=N_1}^{n-1}\gamma_j \\ &\prec k^2M\left\|d\left(x_{N_1}, p_1\right)\right\| + k^2\|v\|M\sum_{j=N_1}^{n-1}\gamma_j \\ &\prec k^2M\frac{\varepsilon}{3k^2M} + k^2\|v\|M\frac{\varepsilon}{6k^2\|v\|M} + k^2M\frac{\varepsilon}{3k^2M} + k^2\|v\|M\frac{\varepsilon}{6k^2\|v\|M} \\ &= \varepsilon\end{aligned}$$

Since ε is an arbitrary positive number, hence $\{x_n\}$ is a Cauchy sequence, therefore it converges to a point, say $p \in C$, and $\lim_{n\to\infty}\|d(x_n, F)\| = 0$ gives that $\|d(x_n, F)\| = 0$. By Lemma 3.1, we know that F is closed. Thus $p \in F$. This completes the proof of the theorem.

If we set $E = \mathbb{R}, P = [0, \infty), d(x, y) = |x - y|, x, y \in \mathbb{R}$, that is $\|.\| = |.|$, we get the following corollary.

Corollary 3.1. Let C be a nonempty closed convex subset of a Banach space X, f and I be same as in Lemma 3.1. Let $F = F(f) \cap F(I) \neq \phi$. For any given $x_1 \in C, \{x_n\}$ is an Ishikawa type iteration scheme with errors defined by

$$x_{n+1} = a_n y_n + b_n I^n y_n + c_n y_n$$

$$y_n = a_n^{'} x_n + b_n^{'} f^n x_n + c_n^{'} v_n, n \geq 1 \tag{3.5}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}, \{c_n^{'}\}$ are real sequences in $[0,1)$ with $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in C. If $\sum_{n=1}^{\infty}(c_n + c_n^{'}) < \infty$, then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $\lim\ \inf_{n\to\infty}\|d(x_n, F)\| = 0$, where

$$d(x_n, F) = \inf\{d(x_n, p): p \in F\}$$

Theorem 3.2. Let X be a convex cone metric space, C be a nonempty closed convex subset of $X, f: C \to C$ be I -asymptotically quasi-nonexpansive mapping with sequence $\{k_n\} \subset [0, \infty)$ and $I: C \to C$ be asymptotically nonexpansive mapping with $\{k_n^{'}\} \subset [0, \infty)$ (without the conditions $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} k_n^{'} < \infty$) and $F = F(f) \cap F(I)$. The Ishikawa type iteration sequence $\{x_n\}$ is generated by (2.1), $\{u_n\}, \{v_n\}$ satisfying (2.2). If $F \neq \phi$ and $\sum_{n=1}^{\infty}(a_n + b_n + c_n) < \infty$, then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $\lim\ \inf_{n\to\infty}\|d(x_n, F)\| = 0$, where $d(x_n, F) = \inf\{d(x_n, p): p \in F\}$.

Proof. For any $p \in F$,

$$\begin{aligned}d(x_{n+1}, p) &= d(W(y_n, I^n y_n, u_n; a_n, b_n, c_n), p) \\ &\preccurlyeq a_n d(y_n, p) + b_n d(I^n y_n, p) + c_n d(u_n, p) \\ &\preccurlyeq a_n d(y_n, p) + b_n(1 + k_n^{'})d(y_n, p) + c_n d(u_n, p)\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
d(y_n, p) &= d(W(x_n, f^n x_n, v_n; a_n', b_n', c_n'), p) \\
&\preccurlyeq a_n' d(x_n, p) + b_n' d(f^n x_n, p) + c_n' d(v_n, p) \\
&\preccurlyeq a_n' d(x_n, p) + b_n'(1 + k_n)(1 + k_n') d(x_n, p) + c_n' d(v_n, p) \qquad (3.7)
\end{aligned}
$$

Substituting (3.7) into (3.6), it can be obtained that

$$
\begin{aligned}
d(x_{n+1}, p) &\preccurlyeq \left(a_n + b_n(1 + k_n')\right)\left[\left(a_n' + b_n'(1 + k_n)(1 + k_n')\right) d(x_n, p) + c_n' d(v_n, p)\right] + c_n d(u_n, p) \\
&\preccurlyeq \left(a_n + b_n(1 + k_n')\right)[\gamma_n d(x_n, p) + c_n' d(v_n, p)] + c_n d(u_n, p) \\
&\preccurlyeq (1 + \alpha_n) d(x_n, p) + \alpha_n v \qquad (3.8)
\end{aligned}
$$

where $\alpha_n = a_n + b_n + c_n, \gamma_n = a_n' + b_n'(1 + k_n)(1 + k_n')$ and $v = (1 + k_n')c_n'\left(d(v_n, p) + d(u_n, p)\right)$.

Since $\{u_n\}, \{v_n\}$ satisfying (2.2), $\sum_{n=1}^{\infty}(a_n + b_n + c_n) < \infty$ for all $n \in \mathbb{N}$, thus $\lim_{n\to\infty} \alpha_n = 0$. By Lemma 2.1, $\lim_{n\to\infty}\|d(x_n, p)\|$ exists for each $p \in F$. Hence Theorem 3.2 can be proven by Theorem 3.1.

If we set $E = \mathbb{R}, P = [0, \infty), d(x, y) = |x - y|, x, y \in \mathbb{R}$, that is $\|.\| = |.|$, we get the following corollary.

Corollary 3.2. Let C be a nonempty closed convex subset of a Banach space X, f and I be same as in Theorem 3.1 and $F = F(f) \cap F(I) \neq \phi$. Let $\{x_n\}$ be an Ishikawa type iteration sequence defined by (3.5). If $\sum_{n=1}^{\infty}(a_n + b_n + c_n) < \infty$, then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $\lim\ \inf_{n\to\infty}\|d(x_n, F)\| = 0$, where $d(x_n, F) = \inf\{d(x_n, p): p \in F\}$.

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Competing Interests

Authors have declared that no competing interests exist.

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