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Fixed Point Theorems in Partially Ordered Rectangular Metric Spaces

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Authors' contributions

This work was carried out in collaboration between all authors. Author AM is a supervisor and authors NY and BM are students of him. Authors AM and NY wrote the first draft of the manuscript. Authors AM and BM managed and edited the manuscript. All authors read and approved the final manuscript.

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Abstract

In the manuscript, a concept of a mixed monotone mapping is acquainted and a coupled fixed point theorems is substantiated for such nonlinear shrinkage mappings in partially ordered exact rectangular metric spaces. We enlarge and universalize the conclusions of these theory.

Keywords: Rectangular metric spaces; Coupled fixed point; Hausdorff spaces; partially ordered set; mixed monotone mapping.

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1 Introduction

Stefan Banach is a Polish mathematician, who established a benchmark fixed point theorem for the Banach Shrinkage Principle (BSP). The BSP has been universalized in many different respects. Many researcher enlarged to the situation of nonlinear shrinkage mappings. Presence of a fixed point in partially ordered metric spaces are imagined in [1-12] so far, here implementations to matrix equations, ordinary differential equations and integral equations are asserted.

A coupled fixed point is acquainted by Bhaskar and Lakshmikantham [2] and some coupled fixed point theorems is demonstrated for mixed monotone mappings in ordered metric spaces. Subsequently, another conclusions on coupled fixed point theory subsist in the theory, for further particulars, the commentator is mentioned to [13,6-12].

Branciari [3] defined a rectangular metric space (RMS) by replacing the sum at the righthand side of the triangle inequality by a three-term expression. He also substantiated an alike of the BSP. The intriguing nature of these spaces has attracted attention, and fixed points theorems for various shrinkage mappings on rectangular metric spaces have been established [14,5,8].

In this article, the presence of a coupled fixed point is attested for a mixed monotone mapping $T: X \times X \to X$ under a universalized shrinkage, build the uniqueness under a supplementary presumption on partially ordered exact rectangular metric space.

We state primary definitions and notations to be used throughout this paper. Rectangular metric spaces are designated as below.

Definition 1.1: [6] Assume X is a not null set and $d: X \times X \to [0, \infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is dissimilar from x and y.

(RM1) $d(x, y) = 0 \Leftrightarrow x = y$,
(RM2) $d(x, y) = d(y, x)$,
(RM3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

At that time the map d is named a rectangular metric and the pair (X, d) is named a rectangular metric spaces (RMS).

Definition 1.2: [10] Suppose $(X, \preceq)$ is a partially ordered set, $T: X \times X \to X$. T has got the mixed monotone property if $T(x, y)$ is monotone nondecreasing with respect to x, is monotone non-increasing with respect to y, namely, for any $x, y \in X$,

$$x_1, x_2, y \in X, x_1 \preceq x_2 \Rightarrow T(x_1, y) \preceq T(x_2, y)$$

and

$$y_1, y_2, x \in X, y_1 \preceq y_2 \Rightarrow T(x, y_1) \succeq T(x, y_2).$$

Definition 1.3: [10] A member $(x, y) \in X \times X$ is named a coupled fixed point of the mapping T if

$$T(x, y) = x \text{ and } T(y, x) = y.$$

For any $(x,y),(u,v)\in X\times X$, the product space $X\times X$ is equipped with the metric ρ described by

$$\rho((x,y),(u,v))=\frac{d(x,u)+d(y,v)}{2}.$$

So ρ is a rectangular metric.

Definition 1.4: [6]

(i) A sequence x_n in a RMS (X,d) is RMS convergent to a limit $x \Leftrightarrow d(x_n,x)\to 0$ as $n\to\infty$.

(ii) A sequence x_n in a RMS (X,d) is RMS Cauchy $\Leftrightarrow$ for every $\varepsilon>0$ there subsist a positive integer $N(\varepsilon)$ with $d(x_n,x_m)<\varepsilon$ for all $n>m>N(\varepsilon)$.

(iii) A RMS (X,d) is named exact if every RMS Cauchy sequence in X is RMS convergent.

The fixed point theorem is described by, Lakzian and Samet [8] as below.

Theorem 1.5: [8] Suppose (X,d) is a Hausdorff and exact RMS and assume $T:X\to X$ is a self-map ensuring

$$\psi(d(Tx,Ty))\le\psi(d(x,y))-\phi(d(x,y))$$

for all $x,y\in X$, and $\psi,\phi\in\Psi$, where ψ is non-decreasing. At that time T has got an individual fixed point in X.

Recently, İ. M. Erhan et al. [6] secured the fabulous conclusion and gave a universalization of Theorem 1.5 for a larger class of (ψ,ϕ) weakly shrinkage mappings.

Theorem 1.6: [6] Suppose (X,d) is a Hausdorff and exact RMS and assume $T:X\to X$ is a self-map convincing

$$\psi(d(Tx,Ty))\le\psi(M(x,y))-\phi(M(x,y))+Lm(x,y)$$

for all $x,y\in X$ and $\psi,\phi\in\Psi$, where $L>0$, the function ψ is nondecreasing and at that time T has got an individual fixed point in X.

$$M(x,y)=max\{d(x,y),d(x,Tx)d(y,Ty)\},$$

$$m(x,y)=min\{d(x,Tx)d(y,Ty),d(x,Ty)d(y,Tx)\}.$$

At that time T has got an individual fixed point in X.

In next section, we give a universalization of Theorem 1.6 for a mixed monotone mappings. We verify a coupled fixed point theorem in partially ordered exact rectangular metric spaces.

2 Main Consequence

Now the main conclusion is certified.

Theorem 2.1: Supposing $(X,\preceq)$ is partially ordered set and (X,d) Hausdorff and exact RMS. Suppose there is a function $\varphi,\theta:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t$, $\theta(t)<t$ and for each $t>0$ $\lim_{r\to t^+}\varphi(r)<t$, $\lim_{r\to t^+}\theta(r)<t$ and also the mixed monotone property on X is verified by $T:X\times X\to X$ be a continuous mapping and

$$d(T(x,y),T(u,v))\leq\varphi(\rho((x,y),(u,v)))-\theta(\rho((x,y),(u,v))) \tag{1}$$

for all $x,y,u,v\in X$ for which $x\preceq u$, $y\succeq v$ There subsist $(x_0,y_0)\in X\times X$ with $x_0\prec T(x_0,y_0)$, $y_0\succeq T(y_0,x_0)$. So T has got an individual coupled fixed point.

Proof. Suppose $x_0,y_0\in X$ is with $x_0\prec T(x_0,y_0)$, $y_0\succeq T(y_0,x_0)$. Let $x_1=T(x_0,y_0)$, $y_1=T(y_0,x_0)$. Then $x_0\prec x_1$, $y_0\succeq y_1$. Again, let $x_2=T(x_1,y_1)$, $y_2=T(y_1,x_1)$. The mixed monotone property is verified by T, obtaining $x_1\prec x_2$, $y_1\succeq y_2$. To continue to do so, two sequences $\{x_n\}$, $\{y_n\}$ are built in X with $x_{n+1}=T(x_n,y_n)$, $y_{n+1}=T(y_n,x_n)$,

$$x_0\prec x_1\prec x_2\prec\cdots\prec x_n\prec x_{n+1}\prec\cdots \tag{2}$$

and,

$$y_0\succeq y_1\succeq y_2\succeq\cdots\succeq y_n\succeq y_{n+1}\succeq\cdots. \tag{3}$$

Denote

$$\delta_n=\rho((x_n,y_n),(x_{n+1},y_{n+1}))=\frac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}. \tag{4}$$

We show that $\delta_n<\delta_{n-1}$. Now, enforcing if the inequality (1) is implemented with $(x,y)=(x_n,y_n)$, $(u,v)=(x_{n-1},y_{n-1})$, for all $n\geq 0$. Utilizing properties of φ, we get

$$\begin{aligned}d(T(x_n,y_n),T(x_{n-1},y_{n-1}))&=d(x_{n+1},x_n)\\&\leq\varphi(\rho((x_n,y_n),(x_{n-1},y_{n-1})))-\theta(\rho((x_n,y_n),(x_{n+1},y_{n+1})))\\&\leq\varphi(\rho((x_n,y_n),(x_{n-1},y_{n-1})))<\rho((x_n,y_n),(x_{n-1},y_{n-1})).\end{aligned} \tag{5}$$

Similarly, we can obtain

$$d(y_{n+1},y_n)<\rho((x_n,y_n),(x_{n-1},y_{n-1})). \tag{6}$$

Thus we acquire

$$\frac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}<\rho((x_n,y_n),(x_{n-1},y_{n-1}).$$

That is $\delta_n<\delta_{n-1}$. Thus $\{\delta_n\}$ is monotone decreasing bounded to the bottom. Hereby, there subsist a $\delta\geq 0$ with

$$\lim_{n\to\infty}\delta_n=\delta.$$

Demonstrating $\delta=0$. Supposing the contrary $\delta>0$. At that time from (1) obtaining

$$\begin{aligned} d(T(x_n,y_n),T(x_{n-1},y_{n-1}))=d(x_{n+1},x_n)&\leq\varphi(\rho((x_n,y_n),(x_{n-1},y_{n-1}))) \\ -\theta(\rho((x_n,y_n),(x_{n+1},y_{n+1})))&\leq\varphi(\rho((x_n,y_n),(x_{n-1},y_{n-1}))). \end{aligned} \tag{7}$$

Similarly, we can obtain

$$d(y_{n+1},y_n)<\varphi(\rho((x_n,y_n),(x_{n-1},y_{n-1}))). \tag{8}$$

Thus we get

$$\frac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}<\rho((x_n,y_n),(x_{n-1},y_{n-1}). \tag{9}$$

While $n\to\infty$ in (9), getting

$$\delta=\lim_{n\to\infty}\delta_n<\lim_{n\to\infty}\delta_{n-1}=\delta.$$

So the incompatibility is obtained. Hence $\delta=0$. Namely

$$\lim_{n\to\infty}\rho((x_n,y_n),(x_{n+1},y_{n+1}))=\lim_{n\to\infty}\frac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}=0.$$

Thus

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0 \text{ and } \lim_{n\to\infty}d(y_n,y_{n+1})=0. \tag{10}$$

Evidencing $\{x_n\}$, $\{y_n\}$ are RMS Cauchy sequences. To accept the opposite that at least one of $\{x_n\}$ or $\{y_n\}$ is not a RMS Cauchy sequence. At that time there subsist an $\varepsilon>0$ when obtaining two subsequences $\{x_{n(i)}\}$ and $\{x_{m(i)}\}$ of $\{x_n\}$ with $n(i)$ is the smallest index where

$$n(i)>m(i)>i,\ d(x_{m(i)},x_{n(i)})+d(y_{m(i)},y_{n(i)})\geq\varepsilon. \tag{11}$$

This means that

$$d(x_{m(i)},x_{n(i)-1})+d(y_{m(i)},y_{n(i)-1})<\varepsilon. \tag{12}$$

By $(\mathrm{RM3})$, we obtain

$$d(x_{m(i)},x_{n(i)})\leq d(x_{m(i)},x_{m(i)-1})+d(x_{m(i)-1},x_{n(i)-1})+d(x_{n(i)-1},x_{n(i)}). \tag{13}$$

Similarly from $(\mathrm{RM3})$, we can obtain

$$d(y_{m(i)},y_{n(i)})\leq d(y_{m(i)},y_{m(i)-1})+d(y_{m(i)-1},y_{n(i)-1})+d(y_{n(i)-1},y_{n(i)}). \tag{14}$$

By adding (13) and (14), from (10), (11) and (12)

$$\begin{aligned}d(x_{m(i)},x_{n(i)})+d(y_{m(i)},y_{n(i)})\leq{}&[d(x_{m(i)},x_{m(i)-1})+d(y_{m(i)},y_{m(i)-1})]+\\&[d(x_{m(i)-1},x_{n(i)-1})+d(y_{m(i)-1},y_{n(i)-1})]+[d(x_{n(i)-1},x_{n(i)})+d(y_{n(i)-1},y_{n(i)})].\end{aligned} \tag{15}$$

Getting the limit as $i\to\infty$ in (15), obtaining by (10), (11)

$$\begin{aligned}\varepsilon&\leq\lim_{i\to\infty}[d(x_{m(i)},x_{n(i)})+d(y_{m(i)},y_{n(i)})]\\&\leq\lim_{i\to\infty}[d(x_{m(i)-1},x_{n(i)-1})+d(y_{m(i)-1},y_{n(i)-1})].\end{aligned} \tag{16}$$

Similarly from $(\mathrm{RM3})$, we can obtain

$$\begin{aligned}d(x_{m(i)-1},x_{n(i)-1})+d(y_{m(i)-1},y_{n(i)-1})\leq{}&[d(x_{m(i)-1},x_{m(i)})+d(y_{m(i)-1},y_{m(i)})]\\&+[d(x_{m(i)},x_{n(i)})+d(y_{m(i)},y_{n(i)})]+[d(x_{n(i)},x_{n(i)-1})+d(y_{n(i)},y_{n(i)-1})].\end{aligned} \tag{17}$$

Having the limit as $i\to\infty$ in (17), we get by (10), (11), (16)

$$\lim_{i\to\infty}[d(x_{m(i)},x_{n(i)})+d(y_{m(i)},y_{n(i)})]=\lim_{i\to\infty}[d(x_{m(i)-1},x_{n(i)-1})+d(y_{m(i)-1},y_{n(i)-1})]. \tag{18}$$

Applying inequality (1) with $(x,y)=(x_{m(i)-1},y_{m(i)-1}),\ (u,v)=(x_{n(i)-1},y_{n(i)-1})$

$$\begin{aligned}d(x_{m(i)},x_{n(i)})&=d(T(x_{m(i)-1},y_{m(i)-1}),T(x_{n(i)-1},y_{n(i)-1}))\\&\leq\varphi(\rho((x_{m(i)-1},y_{m(i)-1}),(x_{n(i)-1},y_{n(i)-1})))\\&<\rho((x_{m(i)-1},y_{m(i)-1}),(x_{n(i)-1},y_{n(i)-1})).\end{aligned} \tag{19}$$

Similarly we get

$$d(y_{m(i)},y_{n(i)})<\rho((x_{m(i)-1},y_{m(i)-1}),(x_{n(i)-1},y_{n(i)-1})). \tag{20}$$

Before by adding (19) and (20) and after taking the limit as $i \to \infty$, we get

$$\lim_{i\to\infty}[d(x_{m(i)}, x_{n(i)}) + d(y_{m(i)}, y_{n(i)})] < \lim_{i\to\infty}[d(x_{m(i)-1}, x_{n(i)-1}) + d(y_{m(i)-1}, y_{n(i)-1})] .$$

From (18), this is a contradiction. Then $\{x_n\},\ \{y_n\}$ are RMS Cauchy sequences. Because (X,d) is exact there subsist $x, y \in X$ with

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y . \tag{21}$$

From continuity of T and since X is Hausdorff, obtaining

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n, y_n) = T(x, y),$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} T(y_n, x_n) = T(y, x).$$

Theorem 2.2: Suppose $(X,\preceq)$ is partially ordered set and (X,d) Hausdorff and exact RMS. Assume $T : X\times X \to X$ is a mapping possessing mixed monotone property. Supposing there is a function φ,θ like in Theorem 2.1 and $\varphi(t) = 0 \Rightarrow t = 0$. Assuming X has property as below:

(a) for all n, if a non-decreasing sequence $\{x_n\} \to x$, at that time $x_n \preceq x$

(b) for all n, if a non-increasing sequence $\{y_n\} \to y$, at that time $y_n \succeq y$.

If there subsist $x_0, y_0 \in X$ with $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$, at that time there subsist $x, y \in X$ with $x = T(x, y),\ y = T(y, x)$, namely T has got an individual coupled fixed point.

Proof. Coming after the proof of Theorem 2.1, constructing a non-decreasing sequence $\{x_n\}$ in X and a non-increasing sequence $\{y_n\}$ in X with $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$ for all $n \geq 0$ and verifying (21).

Thence by properties of X, obtaining $x_n \preceq u$ and $y_n \succeq v$ for all $n \geq 0$. By (1), having

$$\begin{aligned} d(x_{n+1}, T(x, y)) = d(T(x_n, y_n), T(x, y)) &\leq \varphi(\rho((x_n, y_n), (x, y))) \\ &\quad - \theta(\rho((x_n, y_n), (x, y))) \\ &\leq \varphi(\rho((x_n, y_n), (x, y))). \end{aligned}$$

On letting $n \to \infty$, using (21) and properties of φ, we get that

$$\lim_{n\to\infty} d(x_{n+1}, T(x, y)) = 0. \tag{22}$$

Conversely, from (RM3) getting

$$d(x,T(x,y)) \le d(x,x_{n+2}) + d(x_{n+2},x_{n+1}) + d(x_{n+1},T(x,y)).$$

While $n \to \infty$ in the foregoing imparity, utilizing (21), (10) and (22), having $d(x,T(x,y)) = 0$ namely $x = T(x,y)$. Analogous, displaying $y = T(y,x)$.

3 Conclusion

We have specified fixed point theorems in partially ordered rectangular metric. It is interesting that each fixed point theorems are verifying in the theory, because the theory is consisting.

Competing Interests

Authors have declared that no competing interests exist.

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