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A Computation of the Casimir Energy on a Parallelepiped

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The original computations deriving the Casimir energy and force consists of first taking limits of the spectral zeta function and afterwards analytically extending the result. This process of computation presents a weakness in Hendrik Casimir's original argument since limit and analytic continuation do not commute. A case of the Laplacian on a parallelepiped box representing the space as the vacuum between two plates modelled with Dirichlet and periodic Neumann boundary conditions is constructed to address this anomaly. It involves the derivation of the regularised zeta function in terms of the Riemann zeta function on the parallelepiped. The values of the Casimir energy and Casimir force obtained from our derivation agree with those of Hendrik Casimir.

Keywords: Laplacian; zeta function; Casimir energy; parallelepiped; analytic continuation.

MSC class: 35P20; 35R01

1 Introduction

The Casimir force has been a fundamental issue in Quantum Field Theory since the prediction of Hendrik B. G. Casimir and Dirk Polder in the year 1948, [1]. Casimir and Polder in [1] established that there exists a

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force between a pair of neutral perfectly conducting parallel plates placed close together in a vacuum space. Fig. 1 illustrates this idea.

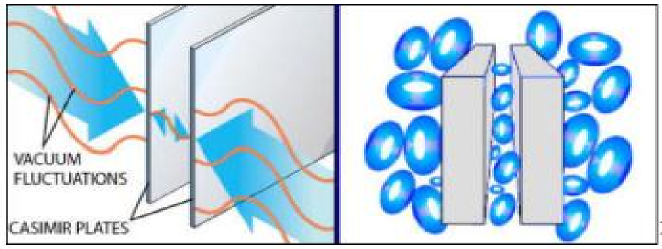

Fig. 1. The conducting plates [2]

Small dielectric bodies interacting at a "reasonable" distance attract, [3], and based on summation of the two-body forces, one may speculate that any two dielectrics would attract. Modern physics indicates that this result derives from the fluctuating electromagnetic waves in a vacuum space, despite the fact that the presence of these waves in a vacuum appears counter intuitive. While this phenomenon has since been reformulated as a pure mathematical problem [4-7] and further described for different Riemannian manifolds using various methods, see for example [3,8-13], the available literature on the mathematical reasoning behind this result is scarce.

Let (M, g) be a closed connected smooth Riemannian manifold, where g is the canonical Riemannian metric on M. The Laplacian on $C^\infty(M)$ is the operator

$$\Delta_g : C^\infty(M) \to C^\infty(M) \tag{1.1}$$

defined in local coordinates by

$$\Delta_g = -\operatorname{div}(\text{grad}) = -\frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}\right) \tag{1.2}$$

in terms of the elements of the metric g. The operator Δ_g extends to a self-adjoint operator on $L^2(M) \supset H^2(M) \to L^2(M)$ with compact resolvent, [14-16], among other literature. This implies that there exists orthonormal basis $\{\psi_k\} \subset L^2(M)$ consisting of eigenfunctions such that

$$\Delta_g \psi_k = \lambda_k \psi_k \tag{1.3}$$

where the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are listed with multiplicities.

The Casimir energy is defined for the Laplacian on the compact Riemannian manifold (M, g), mathematically via the spectral zeta as a function on the set of metrics g on the manifold by $\zeta_g(-\frac{1}{2})$, [17, 8] or via the energy function E_C, [2], given by

$$E_C = \frac{\hbar c}{2}\sum_{n=1}^{\infty}\sqrt{\lambda_n} = \hbar c\sum_{n=1}^{\infty}\lambda_n^{-s} \tag{1.4}$$

at $s = -\frac{1}{2}$. A factor 2 has been introduced for the two possible polarization of the photons.

Of course, the sum in (1.4) does not converge and a regularisation process should be introduced to make it meaningful. Zeta function regularisation assigns a finite value to the divergent series, [3,11,12].

Casimir computed the Casimir energy and force between the plates by first taking the limit of the spectral zeta function as the area of the plates tends to infinity and subsequently computed the analytic continuation of the limit, [18]. The attractive force was shown to be proportional to the fourth power of the inverse of the distance between the plates, [1]. However, this approach is a weakness in his argument since limits and analytic continuation do not commute. The goal of this paper is to address this problem and consequently perform the computation in the standard order. The results obtained agree with the formulae of Casimir and those obtained using other methods.

The paper is organised in sections. Section one is the introduction while section two highlights analytic continuation. Eigenvalues of the Laplacian on the parallelepiped is presented in section three while sections four and five present the main result, namely, the regularised formula for the Casimir energy. Concluding remarks are in section six.

2 Analytic Continuation

Analytic continuation of a given function is a method to extend the domain of the function, i.e find other values of the function which are initially undefined, [19,11]. Given a function f with domain Ω_1, find a function g that matches f on Ω_1 but now defined in a larger region Ω_2. Alternatively, on a complex plane, take a non-singular point x_0 of f with the associated circle of convergence of its Taylor series. This circle of convergence will pass through the nearest singularity of f at x_0. Then take another non-singular point x_1 on this circle of convergence of its Taylor series. Repeat this process. The union of sets of all these circles form a larger region/domain, Ω_3 say, on which f is defined thus $g = (f, \Omega_3)$ forms the analytic continuation of f. Note, if the analytic continuation of f exits, it is unique.

Limits and analytic continuation do not usually commute. To see this consider the following sequence of functions defined for any $z \in \mathbb{C}$ by

$$f_n(z) := \frac{\mathrm{e}^{nz}}{\mathrm{e}^{nz}-1}; \quad \Re(z) > 0.$$

Clearly, $\lim_{n\to\infty} f_n(z) = 1$. The analytic continuation of $\lim_{n\to\infty} f_n(z)$ is the constant function $g_n(z) = 1$ defined on the whole of the complex plane, e.g. for $z = -1$, $g_n(-1) = 1$. However, the analytic continuation of f_n is

$$g_n(z) = \frac{\mathrm{e}^{nz}}{\mathrm{e}^{nz}-1}; \text{for } \Re(z) \neq 0.$$

Now take for instance $z = -1$ we have that $g_n(-1) = \frac{\mathrm{e}^{-n}}{\mathrm{e}^{-n}-1} = \frac{1}{1-\mathrm{e}^{n}}$ whose limit as $n \to \infty$ is 0 (zero). We see that $1 \neq 0$ which shows that the two operations do not commute hence justifying the rationale of this paper.

Limits and integrals; sums and integrals are interchanged in this work using the following theorems.

Theorem 2.1 [19]. *Let* $\sum_{n=0}^{\infty} a_n(x)$ *be uniformly convergent on* $[a,b]$. *Let each* a_n *be continuous on the interval. Then* $\int_a^b \sum_{n=0}^{\infty} a_n(x)\mathrm{d}x = \sum_{n=0}^{\infty} \int_a^b a_n(x)\mathrm{d}x.$

Theorem 2.2 [19]. *(Dominated Convergence Theorem). Let* $\Omega \subseteq \mathrm{R}^n$ *be open and let* $\{\psi_k\}$ *be a sequence of integrable functions on* Ω. *Suppose that* $\lim_{k\to\infty} \psi_k(x) = \psi(x)$ μ *-almost everywhere. Further suppose that there exists* $\omega \geq 0$ *with* $\int_\Omega \omega(x)\mathrm{d}\mu(x) < \infty$ *such that*

$\psi_k(x) \leq \omega(x)\ \forall k.$ Then $\psi(x) \leq \omega(x)$ μ -almost everywhere and

$\lim_{k\to\infty} \int_\Omega \psi_k(x)\mathrm{d}\mu(x) = \int_\Omega \psi(x)\mathrm{d}\mu(x);$ where $\mathrm{d}\mu(x)$ is the measure form on Ω.

Theorem 2.3 [19]. *(Fubini - Tonelli theorem). Let* $\{\psi_k\}$ *be a sequence of measurable functions. Sum and integral such as* $\sum_k \int \psi_k(x)dx$ *can be interchanged in either of the following cases:* $\psi_k \geq 0, \forall k \in \mathrm{N}$ or $\sum_k \int |\psi_k(x)|\, dx < \infty.$

Well known examples of analytic continuation are those of the gamma and Riemann zeta functions, see for example [20,11,12].

From here on, we suppress the subscript g in $\zeta_g(s)$ and Δ_g. We simply write $\zeta(s)$ and Δ for $\zeta_g(s)$ and Δ_g respectively, unless for purpose of emphasis.

3 Eigenvalues of the Laplacian on the Parallelepiped

Consider a region of space, i.e. a box C, defined by $0 < x < l_1$, $0 < y < l_2$, and $0 < z < l_3$ with $l_1, l_2, l_3 \in \mathrm{R}$, enclosing a metallic parallelepiped. The walls of the box are assumed to be non-conducting and they provide the space between the plates of the parallelepiped, (see Fig. 2).

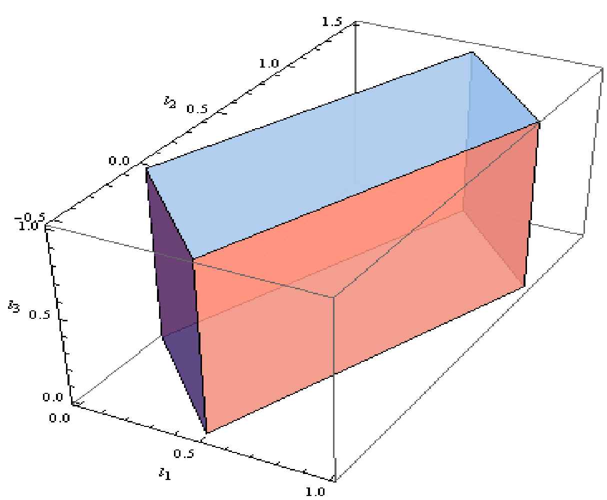

Fig. 2. The parallelepiped in a box C

We make the following definition.

Definition 3.1 [14,15]. *Let* λ_n *be the eigenvalues corresponding to the eigenfunctions* ψ_n *satisfying a Laplacian eigenvalue equation* $\Delta\psi_n = \lambda_n\psi_n$ *subject to specific boundary conditions. The set of all these eigenvalues is called the spectrum of the Laplacian* Δ.

In particular, the eigenstates of the electromagnetic field in the box C are described by the Laplacian eigenvalue problem (LEP):

$$(\Delta-\lambda)\psi = 0 \tag{3.1}$$

where λ and ψ are the associated eigenvalues and eigenfunctions of the Laplacian on smooth functions on the parallelepiped.

In addition, we impose Dirichlet boundary conditions in the x-direction of the plates of the walls of the box because the electric potential vanishes on the plates and periodic boundary condition in the y and z-directions because the plates can be thought-of as a periodic arrangement of finite plates. The boundary conditions will change as the box is compressed. We do not consider what happens outside the box. This choice does not, however, matter since on the whole, the analysis will be independent of this and the Dirichlet boundary conditions could have been chosen instead without changing the end result of the paper. The chosen direction here makes the computations easier to handle.

So the LEP with the boundary conditions is

$$\left.\begin{array}{rcl}(\Delta-\lambda)\psi = 0 & \Leftrightarrow & \dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}+\dfrac{\partial^2\psi}{\partial z^2}-\lambda\psi = 0\\ \text{subject to} & & \\ \psi(0,y,z) & = & \psi(l_1,y,z)=0;\\ \psi(x,0,z) & = & \psi(x,l_2,z);\\ \psi(x,y,0) & = & \psi(x,y,l_3);\\ \dfrac{\partial\psi(x,0,z)}{\partial y} & = & \dfrac{\partial\psi(x,l_2,z)}{\partial y} \quad \text{and}\\ \dfrac{\partial\psi(x,y,0)}{\partial z} & = & \dfrac{\partial\psi(x,y,l_3)}{\partial z}\end{array}\right\} \tag{3.2}$$

By method of separation of variables, we assume non-zero solutions to (3.2) of the form $\psi(x,y,z) = u(x)v(y)w(z)$ so that the from the LEP (3.2), we now consider

$$\left.\begin{array}{rcl}\dfrac{u''}{u}+\dfrac{v''}{v}+\dfrac{w''}{w}+\lambda = 0 & \Leftrightarrow & \dfrac{u''}{u}-\lambda = -\dfrac{v''}{v}-\dfrac{w''}{w}\\ \text{subject to} & & \\ u(0)=u(l_1)=0;\ v(0) & = & v(l_2);\ w(0)=w(l_3);\\ v'(0)=v'(l_2) & \text{and} & w'(0)=w'(l_3)\end{array}\right\} \tag{3.3}$$

So, observe in $\frac{u''}{u}-\lambda = -\frac{v''}{v}-\frac{w''}{w}$ of (3.3) that the left-hand-side depend on x while the right-hand-side depends on y and z. Equality of the sides implies that they must be equal to a constant $\eta \in \mathbb{R}$, say. Hence,

$u'' = -(\lambda - \eta)u$ with $u(0) = u(l_1) = 0$ and

$$-\frac{v''}{v} - \frac{w''}{w} = \eta \Leftrightarrow \eta + \frac{v''}{v} = -\frac{w''}{w} \tag{3.4}$$

which reduces to a second order ordinary differential equation that can be solved quite easiy. It is not difficult to continue this way and obtain the solutions to the LEP (3.2) as

$$\psi_{k,m,n}(x,y,z) = u(x)\cdot v(y)\cdot w(z)$$

$$= (\alpha_k \sin(\frac{k\pi}{l_1}x))\cdot(c_m \sin(\frac{2m\pi}{l_2}y) + d_m \cos(\frac{2m\pi}{l_2}y))$$

$$\cdot(a_n \sin(\frac{2n\pi}{l_3}z) + b_n \cos(\frac{2n\pi}{l_3}z)); \text{ with } k,m,n \in \mathbb{Z}. \tag{3.5}$$

To normalize ψ, we solve $\|\psi\| = 1$. That is

$$\|\psi\|^2 = \|(\alpha_k \sin(\frac{k\pi}{l_1}x))\cdot(c_m \sin(\frac{2m\pi}{l_2}y) + d_m \cos(\frac{2m\pi}{l_2}y))\cdot(a_n \sin(\frac{2n\pi}{l_3}z) + b_n \cos(\frac{2n\pi}{l_3}z))\|^2$$

$$= \int_0^{l_1}\int_0^{l_2}\int_0^{l_2} \alpha_k^2 \sin^2(\frac{k\pi}{l_1}x)\cdot(c_m \sin(\frac{2m\pi}{l_2}y) + d_m \cos(\frac{2m\pi}{l_2}y))^2$$

$$\cdot(a_n \sin(\frac{2n\pi}{l_3}z) + b_n \cos(\frac{2n\pi}{l_3}z))^2 \, dz dy dx$$

$$= \alpha_k^2 (c_m^2 + d_m^2)(a_n^2 + b_n^2)\frac{l_1 l_2 l_3}{8}.$$

So, using that $\|\psi\|^2 = 1$ we have

$$\alpha_k^2 (c_m^2 + d_m^2)(a_n^2 + b_n^2) = \frac{8}{l_1 l_2 l_3}. \tag{3.6}$$

Therefore the spectrum of the Lapacian Δ, subject to the boundary conditions, on the parallelepiped is

$$\lambda_{k,m,n} = \frac{k^2\pi^2}{l_1^2} + \frac{4m^2\pi^2}{l_2^2} + \frac{4n^2\pi^2}{l_3^2} \text{ with } k,m,n \in \mathbb{Z} \tag{3.7}$$

corresponding to the normalised eigenfunction (3.5) satisfying (3.6).

4 The Casimir Energy

Having found the eigenvalues of the Laplacian on the parallelepiped, we proceed to make sense of the divergent sum (1.4), i.e. the Casimir (vacuum) energy between the parallelepiped and the walls of the box, by extending its domain of convergence. Following the method of zeta function regularisation discussed in [3,8-13] among other related literature, given a differential operator A, the generalised zeta function of A is

$$\zeta_A(s) = \mathrm{Tr} A^{-s}. \tag{4.1}$$

If the spectrum of A is discrete, i.e has set of eigenvalues λ_n, we write the spectral zeta function as

$$\zeta_A(s) = \sum_n \frac{1}{\lambda_n^s} \tag{4.2}$$

where the sum is over non-zero eigenvalues. Since the spectrum $\lambda_{k,m,n}$ of the Laplacian is discrete then

$$\zeta_\Delta(s) = \sum_{k,m,n} \frac{1}{\lambda_{k,m,n}^s}. \tag{4.3}$$

So the Casimir energy is obtained on evaluating the total energy on the parallelepiped at $s = -\frac{1}{2}$. That is,

$$E_C = \hbar c \zeta_\Delta(s)|_{s=-\frac{1}{2}}. \tag{4.4}$$

Since $s = -\frac{1}{2}$ is outside the domain of convergence of (4.4), by analytic continuation, we write

$$\zeta_\Delta(s)|_{s=-\frac{1}{2}} = \sum_{k,m,n}^{\infty} \frac{1}{\lambda_{k,m,n}^s}$$

$$= \frac{1}{\Gamma(s)} \sum_{k,m,n}^{\infty} \int_0^\infty t^{s-1} \exp(-t\lambda_{k,m,n}) \mathrm{d}t$$

$$= \frac{1}{\Gamma(s)} \sum_{k,m,n}^{\infty} \int_0^\infty t^{s-1} \exp(-(\frac{k^2\pi^2}{l_1^2} + \frac{4m^2\pi^2}{l_2^2} + \frac{4n^2\pi^2}{l_3^2})t) \mathrm{d}t.$$

By theorem (2.1)

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{k,m,n}^{\infty} \exp(-(\frac{k^2\pi^2}{l_1^2} + \frac{4m^2\pi^2}{l_2^2} + \frac{4n^2\pi^2}{l_3^2})t) \mathrm{d}t.$$

Thus, it suffices to show that $\sum_{k,m,n}^{\infty} \exp(-(\frac{k^2\pi^2}{l_1^2}+\frac{4m^2\pi^2}{l_2^2}+\frac{4n^2\pi^2}{l_3^2})t)$ is uniformly convergent on $[0,\infty)$.

Dominated convergence theorem (2.2) shows that since for

$$\Re(t) > 0, \sum_{k=1}^{\infty} \exp(-\frac{k\pi^2 t}{l_1^2}) < \infty \text{ and } \exp(-\frac{k^2\pi^2 t}{l_1^2}) \leq \exp(-\frac{k\pi^2 t}{l_1^2}) \ \forall k \in \mathbb{Z}^+;$$

the sum $\sum_{k=1}^{\infty} \exp(-\frac{k^2\pi^2 t}{l_1^2})$ converges uniformly for $t \to 0$ and so both

$$\sum_{m=-\infty}^{\infty} \exp(-\frac{4m^2\pi^2 t}{l_2^2}) \text{ and } \sum_{n=-\infty}^{\infty} \exp(-\frac{n^2\pi^2 t}{l_3^2})$$

converge uniformly too.

Our first result is given as theorem (4.1) below.

Theorem 4.1: *The spectral zeta function ζ_Δ of the Laplacian, subject to the Dirichlet and Neumann boundary condition on the parallelepiped admits analytic continuation.*

Proof. Write ζ_Δ as

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \sum_{k,m,n}^{\infty} \exp(-(\frac{k^2\pi^2}{l_1^2}+\frac{4m^2\pi^2}{l_2^2}+\frac{4n^2\pi^2}{l_3^2})t)\, \mathrm{d}t$$

$$+\frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \sum_{k,m,n}^{\infty} \exp(-(\frac{k^2\pi^2}{l_1^2}+\frac{4m^2\pi^2}{l_2^2}+\frac{4n^2\pi^2}{l_3^2})t)\mathrm{d}t.$$

The second integral is holomorphic for all $s \in \mathbb{C}$ while the first integral converges only for $\Re(s) < 1$. So for the first integral define

$$Z_1(s) = \frac{1}{\Gamma(s)} \int_0^1 \Theta_{l_1}(t)\widetilde{\Theta}_{l_2}(t)\widetilde{\Theta}_{l_3}(t) t^{s-1}\mathrm{d}t$$

where

$$\Theta_{l_1}(t) = \sum_{k=1}^{\infty} \exp(-\frac{k^2\pi^2}{l_1^2}t),$$

$$\widetilde{\Theta}_{l_2}(t) = \sum_{m=-\infty}^{\infty} \exp(-\frac{4m^2\pi^2}{l_2^2}t), \text{ and}$$

$$\widetilde{\Theta}_{l_3}(t) = \sum_{n=-\infty}^{\infty} \exp(-\frac{4n^2\pi^2}{l_3^2}t).$$

Firstly note that individually $\Theta_{l_1}(t), \widetilde{\Theta}_{l_2}(t)$ and $\widetilde{\Theta}_{l_3}(t)$ admit asymptotic expansion as $t \to 0$. Also by Fourier transform, if we let $f(x) = \exp(-\frac{\pi^2 x^2}{l_1^2} t)$ then

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-i\omega x) \mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{\pi^2 x^2}{l_1^2} t - i\omega x) \mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-(\frac{\pi x}{l_1^2}\sqrt{t} + i\frac{l_1\omega}{2\pi\sqrt{t}})^2 - \frac{l_1^2\omega^2}{4t\pi^2}) \mathrm{d}x.$$

Now let $y = \frac{\pi x}{l_1^2}\sqrt{t} + i\frac{l_1\omega}{2\pi\sqrt{t}}$ to have

$$\hat{f}(\omega) = \frac{l_1}{\pi\sqrt{2\pi t}} \exp(-\frac{l_1^2\omega^2}{4t\pi^2}) \int_{-\infty}^{\infty} \exp(-y^2) \mathrm{d}y$$

$$= \frac{l_1}{\pi\sqrt{2\pi t}} \exp(-\frac{l_1^2\omega^2}{4t\pi^2})\sqrt{\pi} = \frac{l_1}{\pi\sqrt{2t}} \exp(-\frac{l_1^2\omega^2}{4t\pi^2}).$$

From Poisson summation formula, we deduce that

$$\sum_{k=1}^{\infty} \exp(-\frac{k^2\pi^2}{l_1^2} t) = \sqrt{2\pi} \sum_{\omega=-\infty}^{\infty} \frac{l_1}{\pi\sqrt{2t}} \exp(-\frac{l_1^2\omega^2}{4t\pi^2})$$

$$= \frac{l_1}{\sqrt{\pi t}} \sum_{\omega=-\infty}^{\infty} \exp(-\frac{l_1^2\omega^2}{4t\pi^2})$$

$$= \frac{l_1}{\sqrt{\pi t}} (\sum_{\omega=-\infty}^{-1} \exp(-\frac{l_1^2\omega^2}{4t\pi^2}) + 1 + \sum_{\omega=1}^{\infty} \exp(-\frac{l_1^2\omega^2}{4t\pi^2}))$$

$$= \frac{l_1}{\sqrt{\pi t}} (1 + 2\sum_{\omega=1}^{\infty} \exp(-\frac{l_1^2\omega^2}{4t\pi^2})).$$

Re-arranging terms gives

$$\Theta_{l_1}(t) = \frac{l_1}{\sqrt{\pi t}} \sum_{\omega=1}^{\infty} \exp(-\frac{l_1^2\omega^2}{4\pi^2 t}) + \frac{l_1 - \sqrt{\pi t}}{2\sqrt{\pi t}}.$$

The same method gives

$$\widetilde{\Theta}_{l_2}(t) = \frac{l_2}{\sqrt{\pi t}} \sum_{\nu=1}^{\infty} \exp(-\frac{l_1^2 \nu^2}{16\pi^2 t}) + \frac{l_2}{2\sqrt{\pi t}}, \text{ and}$$

$$\widetilde{\Theta}_{l_3}(t) = \frac{l_3}{\sqrt{\pi t}} \sum_{\xi=1}^{\infty} \exp(-\frac{l_1^2 \xi^2}{16\pi^2 t}) + \frac{l_3}{2\sqrt{\pi t}}.$$

Thus, $\Theta_{l_1}(t)\widetilde{\Theta}_{l_2}(t)\widetilde{\Theta}_{l_3}(t) = \frac{(l_1 - \sqrt{\pi t})l_2 l_3}{8(\pi t)^{3/2}} + \delta(t)$ where $\delta(t)$ are an exponentially damped terms.

Hence,

$$Z_1(s) = \frac{1}{\Gamma(s)} \int_0^1 ([\Theta_{l_1}(t)\widetilde{\Theta}_{l_2}(t)\widetilde{\Theta}_{l_3}(t) - \frac{(l_1 - \sqrt{\pi t})l_2 l_3}{8(\pi t)^{3/2}}] + \frac{(l_1 - \sqrt{\pi t})l_2 l_3}{8(\pi t)^{3/2}}) t^{s-1} \mathrm{d}t$$

$$= \frac{1}{\Gamma(s)} \int_0^1 [\Theta_{l_1}(t)\widetilde{\Theta}_{l_2}(t)\widetilde{\Theta}_{l_3}(t) - \frac{(l_1 - \sqrt{\pi t})l_2 l_3}{8(\pi t)^{3/2}}] t^{s-1} \mathrm{d}t + \frac{l_2 l_3}{8\pi\Gamma(s)} (\frac{l_1}{\sqrt{\pi}(s - \frac{3}{2})} - \frac{1}{s-1})$$

The exponentially damped terms $\delta(t)$ can be computed explicitly and can easily be shown to be holomorphic in s. Therefore, the analytic continuation of ζ_Δ onto the whole s-complex plane is

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^1 [\Theta_{l_1}(t)\widetilde{\Theta}_{l_2}(t)\widetilde{\Theta}_{l_3}(t) - \frac{(l_1 - \sqrt{\pi t})l_2 l_3}{8(\pi t)^{3/2}}] t^{s-1} \mathrm{d}t$$

$$+ \frac{l_2 l_3}{8\pi\Gamma(s)} (\frac{l_1}{\sqrt{\pi}(s - \frac{3}{2})} - \frac{1}{s-1}) + \int_1^{\infty} t^{s-1} \sum_{k,m,n} \exp(-\lambda_{k,m,n} t) \mathrm{d}t \tag{4.5}$$

where $\lambda_{k,m,n}$ is as given before. This proves the meromorphicity of ζ_Δ

5 Evaluating the Casimir Energy

To compute the Casimir energy of the parallelepiped in the vacuum of the box C per unit area, we divide the energy function, E_c, by the area of the parallelepiped, namely, $l_2 l_3$. The fact that this force is only formed when the plates of the walls of the box are close enough together is equivalent to sending the area of the plates to infinity for a fixed distance l_1 between the plates of the walls of the box. Hence, we compute the limit

$$\lim_{l_2, l_3 \to \infty} \frac{\hbar c \zeta_\Delta(s)}{l_2 l_3} |_{s=-1/2}.$$

This leads to the main result of this paper given as the theorem (5.1) below.

Theorem 5.1: *The meromorphic continuation of the spectral zeta function* ζ_Δ *of the Laplacian on the parallelepiped is given by*

$$H_{l_1}(s) = \frac{1}{4\pi(s-1)}(\frac{l_1}{\pi})^{2(s-1)}\zeta_R(2(s-1). \tag{5.1}$$

It has simple poles at $s=1$ and $s=\frac{3}{2}$ only.

This result is clearly confirmed by the mathematica output shown in Fig. 3 which displays the poles of $H_{l_1}(s)$.

Proof. From the analytic continuation of ζ_Δ, we have

$$\frac{\zeta_\Delta(s)}{l_2 l_3} = \frac{1}{\Gamma(s)}\int_0^1 \frac{\delta(t)}{l_2 l_3} t^{s-1}\mathrm{d}t + \frac{1}{8\pi\Gamma(s)}(\frac{l_1}{\sqrt{\pi}(s-\frac{3}{2})} - \frac{1}{s-1}) + \int_1^\infty t^{s-1}\frac{\Theta_{l_1}(t)\widetilde{\Theta}_{l_2}(t)\widetilde{\Theta}_{l_3}(t)}{l_2 l_3}\mathrm{d}t$$

It therefore follows that

$$H_{l_1}(s) = \lim_{l_2,l_3\to\infty}\frac{\zeta_\Delta(s)}{l_2 l_3} = \frac{1}{4\pi\Gamma(s)}[\int_0^1 t^{s-2}\Theta_{l_1}(t)\mathrm{d}t + \int_1^\infty t^{s-2}\Theta_{l_1}(t)]\mathrm{d}t$$

$$= \frac{\Gamma(s-1)}{4\pi\Gamma(s)\Gamma(s-1)}\int_0^\infty t^{s-2}\Theta_{l_1}(t)\mathrm{d}t$$

$$= \frac{1}{4\pi(s-1)}\zeta_{R,l_1}(s-1)$$

since $\frac{\Gamma(s-1)}{\Gamma(s)} = \frac{1}{s-1}$ and where $\zeta_{R,l_1}(s-1) := \frac{1}{\Gamma(s-1)}\int_0^\infty t^{s-2}\Theta_{l_1}(t)\mathrm{d}t$ is the Riemann zeta function dependent on l_1.

Now since

$$\zeta_{R,l_1}(s-1) = \sum_{k=1}^\infty (\frac{k^2\pi^2}{l_1^2})^{-s} = (\frac{\pi}{l_1})^{-2s}\sum_{k=1}^\infty k^{-2s}$$

$$= (\frac{l_1}{\pi})^{2s}\zeta_R(2s)$$

we have

$$H_{l_1}(s) = \frac{1}{4\pi(s-1)}(\frac{l_1}{\pi})^{2(s-1)}\zeta_R(2(s-1)$$

proving the theorem.

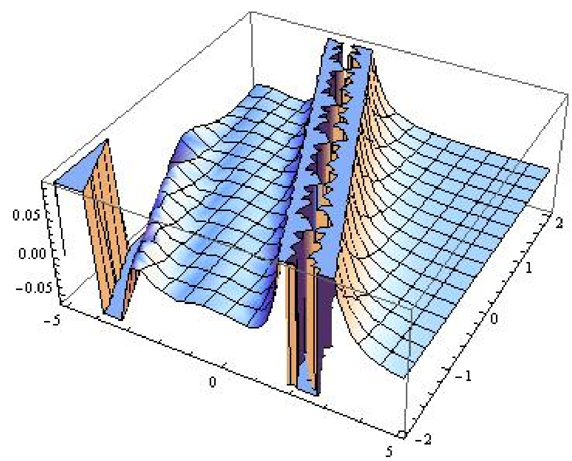

Fig. 3. The analytic continuity of $H_{l_1}(s)$

From equation (5.1) the Casimir energy is read-off as

$$H_{l_1}(-\frac{1}{2}) = -\frac{1}{6\pi}(\frac{l_1}{\pi})^{-3}\zeta_R(-3) = -\frac{\pi^2}{720 l_1^3}.$$

Consequently, the energy, E_c, per unit area between the two plates of the walls of the box C and the parallelepiped is

$$E_c = -\frac{\hbar c \pi^2}{720 l_1^3}$$

and the force per unit area F_c is

$$F_c = \frac{\mathrm{d}E_c}{\mathrm{d}l_1} = -\frac{\hbar c \pi^2}{240 l_1^4}$$

which agree with the original values of Casimir; [21].

6 Conclusion

This paper has successfully derived, in the right mathematical order, the Casimir energy and force as originally found by Casimir. The computations were made less-complicated in this work by imposing

Dirichlet boundary condition in the x -direction and periodic Neumann boundary condition in the y - and z -directions of the box hosting the parallelepiped in the vacuum.

Specifically, we showed that finding the analytic continuation of the spectral zeta function and then performing limit is the mathematical sound thing to do in computing Casimir energy of the Lapacian on closed Riemannian manifolds such as the parallelepiped.

Competing Interests

Author has declared that no competing interests exist.

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