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On Summing Formulas for Horadam Numbers

Yüksel Soykan[1*]

[1] *Department of Mathematics, Faculty of Art and Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the summation formulas for generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas; summing formulas.

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1 INTRODUCTION

Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = c,\ W_1 = d, \quad (n \geq 2) \tag{1.1}$$

**Corresponding author: E-mail: yuksel_soykan@hotmail.com;*

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of c, d, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	Notation: $W_n(c, d; r, s)$	OEIS: [5]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 1.1. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the generalized Fibonacci numbers) If $r + s - 1 \neq 0$, then*

$$\sum_{i=0}^{n} W_i = \frac{W_{n+2} + (1-r)W_{n+1} - W_1 + (r-1)W_0}{r+s-1}.$$

(b) *If $(r - s + 1)(r + s - 1) \neq 0$ then*

$$\sum_{i=0}^{n} W_{2i} = \frac{(1-s)W_{2n+2} + rsW_{2n+1} + (s-1)W_2 - rsW_1 + (r^2 - s^2 + 2s - 1)W_0}{(r-s+1)(r+s-1)}.$$

(c) *If $(r - s + 1)(r + s - 1) \neq 0$ then*

$$\sum_{i=0}^{n} W_{2i+1} = \frac{rW_{2n+2} + (s - s^2)W_{2n+1} - rW_2 + (r^2 + s - 1)W_1}{(r-s+1)(r+s-1)}.$$

Proof. This is given in [6].

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 1.2. *For $n \geq 1$ we have the following formulas:*

(a) *(Sum of the generalized Fibonacci numbers with negative indices) If $r + s - 1 \neq 0$, then*

$$\sum_{i=1}^{n} W_{-i} = \frac{-(r+s)W_{-n-1} - sW_{-n-2} + W_1 + (1-r)W_0}{r+s-1}.$$

(b) *If* $(r-s+1)(r+s-1)\neq 0$ *then*

$$\sum_{i=1}^{n} W_{-2i} = \frac{(s-1)W_{-2n} - rsW_{-2n-1} + rW_1 + (1-s-r^2)W_0}{(r-s+1)(r+s-1)}.$$

(c) *If* $(r-s+1)(r+s-1)\neq 0$ *then*

$$\sum_{i=1}^{n} W_{-2i+1} = \frac{-rW_{-2n} + (s^2-s)W_{-2n-1} + (1-s)W_1 + rsW_0}{(r-s+1)(r+s-1)}.$$

Proof. This is given in [6].

In this work, we investigate some summation formulas of generalized Fibonaci numbers. We present some works on summing formulas of the numbers in the following Table 2.

Table 2. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[7], [8], [9]
Generalized Fibonacci	[10],[6]
Generalized Tribonacci	[11],[12],[13],[14]
Generalized Tetranacci	[15],[16], [17]
Generalized Pentanacci	[18],[19]
Generalized Hexanacci	[20]

2 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. *For* $n\geq 0$ *we have the following formulas:*

(a) *If* $r+s-1\neq 0,$ *then*

$$\sum_{i=0}^{n} iW_i = \frac{\Delta_1}{(r+s-1)^2}$$

where

$$\begin{aligned}\Delta_1 &= ((r-2)+(s+r-1)n)W_{n+2} + ((-r^2+2r-s-1)-(r-1)(r+s-1)n)W_{n+1} \\ &\quad +(1+s)W_1 + s(2-r)W_0.\end{aligned}$$

(b) *If* $(r-s+1)(r+s-1)\neq 0$ *then*

$$\sum_{i=0}^{n} iW_{2i} = \frac{\Delta_2}{(r-s+1)^2(r+s-1)^2}$$

where

$$\begin{aligned}\Delta_2 &= ((1-s)(r-s+1)(r+s-1)n - (r^2s+s^2-2s+1))W_{2n+2} \\ &\quad +rs((r-s+1)(r+s-1)n + r^2+2s-2)W_{2n+1} \\ &\quad +r(1-s^2)W_1 + s(r^2s+s^2-2s+1)W_0.\end{aligned}$$

(c) *If* $(r-s+1)(r+s-1)\neq 0$ *then*

$$\sum_{i=0}^{n} iW_{2i+1} = \frac{\Delta_3}{(r-s+1)^2(r+s-1)^2}$$

where

$$\begin{aligned}\Delta_3 &= r((r-s+1)(r+s-1)n+s^2-1)W_{2n+2} \\ &+(s(1-s)(r+s-1)(r-s+1)n-s(r^2s+s^2-2s+1))W_{2n+1} \\ &+(s^3+r^2-2s^2+s)W_1+rs(1-s^2)W_0.\end{aligned}$$

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}$$

i.e.

$$sW_{n-2} = W_n - rW_{n-1}$$

we obtain

$$\begin{aligned} snW_n &= nW_{n+2}-rnW_{n+1} \\ s(n-1)W_{n-1} &= (n-1)W_{n+1}-r(n-1)W_n \\ s(n-2)W_{n-2} &= (n-2)W_n-r(n-2)W_{n-1} \\ s(n-3)W_{n-3} &= (n-3)W_{n-1}-r(n-3)W_{n-2} \\ &\vdots \\ s5W_5 &= 5W_7-r5W_6 \\ s4W_4 &= 4W_6-r4W_5 \\ s3W_3 &= 3W_5-r3W_4 \\ s2W_2 &= 2W_4-r2W_3 \\ sW_1 &= W_3-rW_2 \end{aligned}$$

If we add the equations by side by, we get

$$s\sum_{i=0}^{n} iW_i = \sum_{i=3}^{n+2}(i-2)W_i - r\sum_{i=2}^{n+1}(i-1)W_i \tag{2.1}$$

for $n \geq 0$. Note that

$$\begin{aligned}\sum_{i=3}^{n+2}(i-2)W_i &= W_1+2W_0+(n-1)W_{n+1}+nW_{n+2}+\sum_{i=0}^{n} iW_i - 2\sum_{i=0}^{n} W_i \\ \sum_{i=2}^{n+1}(i-1)W_i &= W_0+nW_{n+1}+\sum_{i=0}^{n} iW_i - \sum_{i=0}^{n} W_i.\end{aligned}$$

If we put them in (2.1) then it follows that

$$(r+s-1)\sum_{i=0}^{n} iW_i = W_1+2W_0-rW_0+(n-rn-1)W_{n+1}+nW_{n+2}+(r-2)\sum_{i=0}^{n} W_i$$

for $n \geq 0$. Then, if we use Theorem 1.1 (a), the required results of (a) follows.

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we obtain

$$\begin{array}{rcl} rnW_{2n+1} & = & nW_{2n+2} - snW_{2n} \\ r(n-1)W_{2n-1} & = & (n-1)W_{2n} - s(n-1)W_{2n-2} \\ & & \vdots \\ r4W_9 & = & 4W_{10} - s4W_8 \\ r3W_7 & = & 3W_8 - s3W_6 \\ r2W_5 & = & 2W_6 - s2W_4 \\ rW_3 & = & W_4 - sW_2 \\ r \times 0 \times W_1 & = & 0 \times W_2 - s \times 0 \times W_0 \end{array}$$

Now, if we add the above equations by side by, we get

$$r\sum_{i=0}^{n} iW_{2i+1} = \sum_{i=1}^{n+1}(i-1)W_{2i} - s\sum_{i=0}^{n} iW_{2i} \tag{2.2}$$

for $n \geq 0$. Note that

$$\sum_{i=1}^{n+1}(i-1)W_{2i} = W_0 + nW_{2n+2} + \sum_{i=0}^{n}(i-1)W_{2i} = W_0 + nW_{2n+2} + \sum_{i=0}^{n} iW_{2i} - \sum_{i=0}^{n} W_{2i}.$$

If we put this in (2.2) we obtain

$$r\sum_{i=0}^{n} iW_{2i+1} = W_0 + nW_{2n+2} + (1-s)\sum_{i=0}^{n} iW_{2i} - \sum_{i=0}^{n} W_{2i} \tag{2.3}$$

for $n \geq 0$. Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} \Rightarrow rW_n = W_{n+1} - sW_{n-1} \Rightarrow rW_{2n} = W_{2n+1} - sW_{2n-1}$$

we write the following obvious equations;

$$\begin{array}{rcl} r(n+1)W_{2n+2} & = & (n+1)W_{2n+3} - s(n+1)W_{2n+1} \\ rnW_{2n} & = & nW_{2n+1} - snW_{2n-1} \\ r(n-1)W_{2n-2} & = & (n-1)W_{2n-1} - s(n-1)W_{2n-3} \\ & & \vdots \\ r4W_8 & = & 4W_9 - s4W_7 \\ r3W_6 & = & 3W_7 - s3W_5 \\ r2W_4 & = & 2W_5 - s2W_3 \\ rW_2 & = & W_3 - sW_1 \\ r \times 0 \times W_0 & = & 0 \times W_1 - s \times 0 \times W_{-1} \end{array}$$

Now, if we add the above equations by side by, we obtain

$$r\sum_{i=0}^{n} iW_{2i} = \sum_{i=0}^{n} iW_{2i+1} - s\sum_{i=-1}^{n-1}(i+1)W_{2i+1} \tag{2.4}$$

for $n \geq 0$. Note that

$$\sum_{i=-1}^{n-1}(i+1)W_{2i+1} = -(n+1)W_{2n+1} + \sum_{i=0}^{n} iW_{2i+1} + \sum_{i=0}^{n} W_{2i+1}$$

If we put this in (2.4) we obtain

$$r\sum_{i=0}^{n} iW_{2i} = (1-s)\sum_{i=0}^{n} iW_{2i+1} + s(n+1)W_{2n+1} - s\sum_{i=0}^{n} W_{2i+1} \tag{2.5}$$

for $n \geq 0$. Then, using Theorem 1.1 (b) and (c) and

$$W_2 = (rW_1 + sW_0)$$

and solving the system (2.3)-(2.5), the required result of (b) and (c) follow.

Taking $r = s = 1$ in Theorem 2.1 (a), (b) and (c) we obtain the following proposition.

Proposition 2.1. *If $r = s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{i=0}^{n} iW_i = (n-1)\, W_{n+2} - W_{n+1} + 2W_1 + W_0$.

(b) $\sum_{i=0}^{n} iW_{2i} = -W_{2n+2} + (n+1)\, W_{2n+1} + W_0$.

(c) $\sum_{i=0}^{n} iW_{2i+1} = nW_{2n+2} - W_{2n+1} + W_1$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.2. *For $n \geq 0$, Fibonacci numbers have the following properties:*

(a) $\sum_{i=0}^{n} iF_i = (n-1)\, F_{n+2} - F_{n+1} + 2$.

(b) $\sum_{i=0}^{n} iF_{2i} = -F_{2n+2} + (n+1)\, F_{2n+1}$.

(c) $\sum_{i=0}^{n} iF_{2i+1} = nF_{2n+2} - F_{2n+1} + 1$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.3. *For $n \geq 0$, Lucas numbers have the following properties:*

(a) $\sum_{i=0}^{n} iL_i = (n-1)\, L_{n+2} - L_{n+1} + 4$.

(b) $\sum_{i=0}^{n} iL_{2i} = -L_{2n+2} + (n+1)\, L_{2n+1} + 2$.

(c) $\sum_{i=0}^{n} iL_{2i+1} = nL_{2n+2} - L_{2n+1} + 1$.

Taking $r = 2, s = 1$ in Theorem 2.1 (a), (b) and (c) we obtain the following proposition.

Proposition 2.2. *If $r = 2, s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{i=0}^{n} iW_i = \frac{1}{2}(nW_{n+2} - (1+n)W_{n+1} + W_1)$.

(b) $\sum_{i=0}^{n} iW_{2i} = \frac{1}{4}(-W_{2n+2} + 2(n+1)W_{2n+1} + W_0)$.

(c) $\sum_{i=0}^{n} iW_{2i+1} = \frac{1}{4}(2nW_{2n+2} - W_{2n+1} + W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.4. *For $n \geq 0$, Pell numbers have the following properties:*

(a) $\sum_{i=0}^{n} iP_i = \frac{1}{2}(nP_{n+2} - (1+n)P_{n+1} + 1)$.

(b) $\sum_{i=0}^{n} iP_{2i} = \frac{1}{4}(-P_{2n+2} + 2(n+1)P_{2n+1})$.

(c) $\sum_{i=0}^{n} iP_{2i+1} = \frac{1}{4}(2nP_{2n+2} - P_{2n+1} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.5. *For $n \geq 0$, Pell-Lucas numbers have the following properties:*

(a) $\sum_{i=0}^{n} iQ_i = \frac{1}{2}(nQ_{n+2} - (1+n)Q_{n+1} + 2)$.

(b) $\sum_{i=0}^{n} iQ_{2i} = \frac{1}{4}(-Q_{2n+2} + 2(n+1)Q_{2n+1} + 2)$.

(c) $\sum_{i=0}^{n} iQ_{2i+1} = \frac{1}{4}(2nQ_{2n+2} - Q_{2n+1} + 2)$.

If $r = 1, s = 2$ then $(r - s + 1)(r + s - 1) = 0$ so we can't use Theorem 2.1 (b) and (c), directly. However, we can find $\sum_{i=0}^{n} iW_{2i}$ and $\sum_{i=0}^{n} iW_{2i+1}$ using mathematical induction which is given in the following theorem.

Theorem 2.6. *If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{i=0}^{n} iW_i = \frac{1}{4}((2n-1)W_{n+2} - 2W_{n+1} + 3W_1 + 2W_0)$.

(b) $\sum_{i=0}^{n} iW_{2i} = \frac{1}{54}((4+21n)W_{2n+2} - 2(10+3n)W_{2n+1} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)n^2)$.

(c) $\sum_{i=0}^{n} iW_{2i+1} = \frac{1}{54}((8+15n)W_{2n+2} + 2(-20+21n)W_{2n+1} + 16(2W_1 - W_0) + 9(W_1 - 2W_0)n^2)$.

Proof. (b) and (c) can be proved by mathematical induction.

(a) Taking $r = 1, s = 2$ in Theorem 2.1 (a) we obtain (a).

(b) The proof will be by induction on n. Before the proof, we recall some information on generalized Jacobsthal numbers. A generalized Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} + 2W_{n-2}; \;\; W_0 = a, \; W_1 = b, \quad (n \geq 2) \tag{2.6}$$

with the initial values W_0, W_1 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.6) holds for all integer n. The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 3.

Table 3. A few generalized Jacobsthal numbers

n	W_n	W_{-n}
0	W_0	...
1	W_1	$-\frac{1}{2}W_0 + \frac{1}{2}W_1$
2	$2W_0 + W_1$	$\frac{3}{4}W_0 - \frac{1}{4}W_1$
3	$2W_0 + 3W_1$	$-\frac{5}{8}W_0 + \frac{3}{8}W_1$
4	$6W_0 + 5W_1$	$\frac{11}{16}W_0 - \frac{5}{16}W_1$
5	$10W_0 + 11W_1$	$-\frac{21}{32}W_0 + \frac{11}{32}W_1$
6	$22W_0 + 21W_1$	$\frac{43}{64}W_0 - \frac{21}{64}W_1$

Binet formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as

$$t^2 - t - 2 = 0.$$

The roots of characteristic equation are

$$\alpha = 2, \;\; \beta = -1$$

and the roots satisfy the following

$$\alpha + \beta = 1, \; \alpha\beta = -2, \; \alpha - \beta = 3.$$

Using these roots and the recurrence relation, Binet formula can be given as

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} = \frac{A \times 2^n - B(-1)^n}{3} \tag{2.7}$$

where $A = W_1 - W_0\beta = W_1 + W_0$ and $B = W_1 - W_0\alpha = W_1 - 2W_0$.
We now prove (b) by induction on n. If $n = 1$ we that the sum formula reduces to the relation

$$W_2 = \frac{1}{54}((4 + 21 \times 1)W_4 - 2(10 + 3 \times 1)W_3 + 8(2W_1 - W_0) - 9(W_1 - 2W_0)). \tag{2.8}$$

Since

$$\begin{aligned} W_2 &= (2W_0 + W_1), \\ W_3 &= (2W_0 + 3W_1), \\ W_4 &= (6W_0 + 5W_1), \end{aligned}$$

(2.8) is true. Assume that the relation in (b) is true for $n = m$, i.e.,

$$\sum_{i=1}^{m} iW_{2i} = \frac{1}{54}((4 + 21m)W_{2m+2} - 2(10 + 3m)W_{2m+1} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)m^2)$$

Then we get

$$\begin{aligned} \sum_{i=1}^{m+1} iW_{2i} &= (m+1)W_{2m+2} + \sum_{i=1}^{m} iW_{2i} \\ &= (m+1)W_{2m+2} + \frac{1}{54}((4 + 21m)W_{2m+2} - 2(10 + 3m)W_{2m+1} \\ &\quad +8(2W_1 - W_0) - 9(W_1 - 2W_0)m^2) \\ &= \frac{1}{54}((58 + 75m)W_{2m+2} - 2(10 + 3m)W_{2m+1} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)m^2) \\ &= \frac{1}{54}((58 + 75m)W_{2m+2} - 2(10 + 3m)W_{2m+1} + 9(W_1 - 2W_0)(1 + 2m) \\ &\quad +8(2W_1 - W_0) - 9(W_1 - 2W_0)(m+1)^2) \\ &= \frac{1}{54}((4 + 21(m+1))W_{2m+4} - 2(10 + 3(m+1))W_{2m+3} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)(m+1)^2) \\ &= \frac{1}{54}((4 + 21(m+1))W_{2(m+1)+2} - 2(10 + 3(m+1))W_{2(m+1)+1} \\ &\quad +8(2W_1 - W_0) - 9(W_1 - 2W_0)(m+1)^2) \end{aligned}$$

where

$$(58+75m)W_{2m+2} - 2(10+3m)W_{2m+1} + 27(W_1 - 2W_0) = (4+21(m+1))W_{2m+4} - 2(10+3(m+1))W_{2m+3}. \tag{2.9}$$

Note that (2.9) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n = m + 1$.

(c) We now prove (c) by induction on n. If $n=1$ we see that the sum formula reduces to the relation

$$W_3 = \frac{1}{54}((8+15)W_4 + 2(-20+21)W_3 + 16(2W_1 - W_0) + 9(W_1 - 2W_0)). \tag{2.10}$$

Since

$$\begin{aligned} W_3 &= (2W_0 + 3W_1) \\ W_4 &= (6W_0 + 5W_1) \end{aligned}$$

(2.10) is true. Assume that the relation in (c) is true for $n=m$, i.e.,

$$\sum_{i=0}^{m} iW_{2i+1} = \frac{1}{54}((8+15m)W_{2m+2} + 2(-20+21m)W_{2m+1} + 16(2W_1 - W_0) + 9(W_1 - 2W_0)m^2)$$

Then we get

$$\begin{aligned} \sum_{i=0}^{m+1} iW_{2i+1} &= (m+1)W_{2m+3} + \sum_{i=0}^{m} iW_{2i+1} \\ &= (m+1)W_{2m+3} + \frac{1}{54}((8+15m)W_{2m+2} + 2(-20+21m)W_{2m+1} \\ &\quad +16(2W_1 - W_0) + 9(W_1 - 2W_0)m^2) \\ &= \frac{1}{54}(54(m+1)W_{2m+3} + (8+15m)W_{2m+2} + 2(-20+21m)W_{2m+1} - 9(2m+1)(W_1 - 2W_0) \\ &\quad +16(2W_1 - W_0) + 9(W_1 - 2W_0)(m+1)^2) \\ &= \frac{1}{54}((8+15(m+1))W_{2m+4} + 2(-20+21(m+1))W_{2m+3} \\ &\quad +16(2W_1 - W_0) + 9(W_1 - 2W_0)(m+1)^2) \\ &= \frac{1}{54}((8+15(m+1))W_{2(m+1)+2} + 2(-20+21(m+1))W_{2(m+1)+1} \\ &\quad +16(2W_1 - W_0) + 9(W_1 - 2W_0)(m+1)^2) \end{aligned}$$

where

$$\begin{aligned} & 54(m+1)W_{2m+3} + (8+15m)W_{2m+2} + 2(-20+21m)W_{2m+1} - 9(2m+1)(W_1 - 2W_0) \\ = \; & (8+15(m+1))W_{2m+4} + 2(-20+21(m+1))W_{2m+3}. \end{aligned} \tag{2.11}$$

(2.11) can be proved by using Binet formula of W_n. Hence, the relation in (c) holds also for $n=m+1$.

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.7. *For $n \geq 0$, Jacobsthal numbers have the following property:*

(a) $\sum_{i=0}^{n} iJ_i = \frac{1}{4}((2n-1)J_{n+2} - 2J_{n+1} + 3)$.

(b) $\sum_{i=0}^{n} iJ_{2i} = \frac{1}{54}((4+21n)J_{2n+2} - 2(10+3n)J_{2n+1} + 16 - 9n^2)$.

(c) $\sum_{i=0}^{n} iJ_{2i+1} = \frac{1}{54}((8+15n)J_{2n+2} + 2(-20+21n)J_{2n+1} + 32 + 9n^2)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.8. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{i=0}^{n} ij_i = \frac{1}{4}((2n-1)j_{n+2} - 2j_{n+1} + 7)$.

(b) $\sum_{i=0}^{n} ij_{2i} = \frac{1}{54}((4+21n)j_{2n+2} - 2(10+3n)j_{2n+1} + 27n^2)$.

(c) $\sum_{i=0}^{n} ij_{2i+1} = \frac{1}{54}((8+15n)j_{2n+2} + 2(-20+21n)j_{2n+1} - 27n^2)$.

3 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas:*

(a) *If $r+s-1 \neq 0,$ then*

$$\sum_{i=1}^{n} iW_{-i} = \frac{\Delta_4}{(r+s-1)^2}$$

where

$$\begin{aligned} \Delta_4 &= -s\left((r+s-1)n+r+2s\right)W_{-n-2} - ((r+s-1)(r+s)n \\ &\quad +(r+s)^2+s)W_{-n-1} + (1+s)W_1 + s(2-r)W_0. \end{aligned}$$

(b) *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=1}^{n} iW_{-2i} = \frac{\Delta_5}{(r-s+1)^2(r+s-1)^2}$$

where

$$\begin{aligned} \Delta_5 &= ((s-1)(r-s+1)(r+s-1)n - (s^3+r^2-2s^2+s))W_{-2n} \\ &\quad +rs(-(r-s+1)(r+s-1)n+s^2-1)W_{-2n-1} \\ &\quad +r(1-s^2)W_1 + s(r^2s+s^2-2s+1)W_0. \end{aligned}$$

(c) *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=1}^{n} iW_{-2i+1} = \frac{\Delta_6}{(r-s+1)^2(r+s-1)^2}$$

where

$$\begin{aligned} \Delta_6 &= (-r(r-s+1)(r+s-1)n + (2s^2-r^2-2s)r)W_{-2n} \\ &\quad +s((s-1)(r-s+1)(r+s-1)n+2s^2-r^2-s^3-s)W_{-2n-1} \\ &\quad +(s^3+r^2-2s^2+s)W_1 + rs(1-s^2)W_0. \end{aligned}$$

Proof.

(a) Using the recurrence relation

$$\begin{aligned} W_{-n+2} &= r \times W_{-n+1} + s \times W_{-n} \\ &\Rightarrow W_{-n} = -\frac{r}{s} \times W_{-n+1} + \frac{1}{s}W_{-n+2} \\ &\Rightarrow W_{-n} = -\frac{r}{s} \times W_{-(n-1)} + \frac{1}{s}W_{-(n-2)} \end{aligned}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

or

$$W_{-n} = \frac{1}{s}W_{-n+2} - \frac{r}{s}W_{-n+1}$$

we obtain

$$\begin{aligned}
s\times(n+2)\times W_{-n-2} &= (n+2)\times W_{-n}-r\times(n+2)\times W_{-n-1}\\
s\times(n+1)\times W_{-n-1} &= (n+1)\times W_{-n+1}-r\times(n+1)\times W_{-n}\\
s\times n\times W_{-n} &= n\times W_{-n+2}-r\times n\times W_{-n+1}\\
s\times(n-1)\times W_{-n+1} &= (n-1)\times W_{-n+3}-r\times(n-1)\times W_{-n+2}\\
s\times(n-2)\times W_{-n+2} &= (n-2)\times W_{-n+4}-r\times(n-2)\times W_{-n+3}\\
&\vdots\\
s\times 5\times W_{-5} &= 5\times W_{-3}-r\times 5\times W_{-4}\\
s\times 4\times W_{-4} &= 4\times W_{-2}-r\times 4\times W_{-3}\\
s\times 3\times W_{-3} &= 3\times W_{-1}-r\times 3\times W_{-2}\\
s\times 2\times W_{-2} &= 2\times W_{0}-r\times 2\times W_{-1}\\
s\times 1\times W_{-1} &= 1\times W_{1}-r\times 1\times W_{0}.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
s((n+1)W_{-n-1}+(n+2)W_{-n-2}+\sum_{i=1}^{n} iW_{-i}) &= (W_1+2W_0+\sum_{i=1}^{n}(i+2)W_{-i})\\
-r((n+2)W_{-n-1}+W_0+\sum_{i=1}^{n}(i+1)W_{-i})
\end{aligned}$$

for $n\geq 1$. Note that since

$$\begin{aligned}
\sum_{i=1}^{n}(i+2)W_{-i} &= \sum_{i=1}^{n} iW_{-i}+2\sum_{i=1}^{n}W_{-i},\\
\sum_{i=1}^{n}(i+1)W_{-i} &= \sum_{i=1}^{n} iW_{-i}+\sum_{i=1}^{n}W_{-i},
\end{aligned}$$

we have

$$\begin{aligned}
& s((n+1)W_{-n-1}+(n+2)W_{-n-2}+\sum_{i=1}^{n} iW_{-i})\\
=\; & (W_1+2W_0+(\sum_{i=1}^{n} iW_{-i}+2\sum_{i=1}^{n}W_{-i}))-r((n+2)W_{-n-1}+W_0+(\sum_{i=1}^{n} iW_{-i}+\sum_{i=1}^{n}W_{-i}))
\end{aligned}$$

for $n\geq 1$. Then, using Theorem 1.1 (a), the required results of (a) follows.

(b) and (c) Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$rW_{-n+1}=W_{-n+2}-sW_{-n}$$

we obtain

$$\begin{array}{rcl} r\times(n+1)\times W_{-2n-1} & = & (n+1)\times W_{-2n}-s\times(n+1)\times W_{-2n-2} \\ r\times n\times W_{-2n+1} & = & n\times W_{-2n+2}-s\times n\times W_{-2n} \\ r\times(n-1)\times W_{-2n+3} & = & (n-1)\times W_{-2n+4}-s\times(n-1)\times W_{-2n+2} \\ r\times(n-2)\times W_{-2n+5} & = & (n-2)\times W_{-2n+6}-s\times(n-2)\times W_{-2n+4} \\ & \vdots & \\ r\times 3\times W_{-5} & = & 3\times W_{-4}-s\times 3\times W_{-6} \\ r\times 2\times W_{-3} & = & 2\times W_{-2}-s\times 2\times W_{-4} \\ r\times 1\times W_{-1} & = & 1\times W_{0}-s\times 1\times W_{-2}. \end{array}$$

If we add the equations by side by, we get

$$r\sum_{i=1}^{n} iW_{-2i+1}=(-(n+1)W_{-2n}+W_{0}+\sum_{i=1}^{n}(i+1)W_{-2i})-s\sum_{i=1}^{n} iW_{-2i}$$

for $n\geq 1$. Since

$$\sum_{i=1}^{n}(i+1)W_{-2i}=\sum_{i=1}^{n} iW_{-2i}+\sum_{i=1}^{n} W_{-2i}$$

it follows that

$$r\sum_{i=1}^{n} iW_{-2i+1}=-(n+1)W_{-2n}+W_{0}+(1-s)\sum_{i=1}^{n} iW_{-2i}+\sum_{i=1}^{n} W_{-2i} \qquad (3.1)$$

for $n\geq 1$. Similarly, using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$\begin{array}{rcl} rW_{-n+1} & = & W_{-n+2}-sW_{-n}\Rightarrow rW_{-2n+1}=W_{-2n+2}-sW_{-2n} \\ & \Rightarrow & rW_{-2n+1-1}=W_{-2n+2-1}-sW_{-2n-1} \\ & \Rightarrow & rW_{-2n}=W_{-2n+1}-sW_{-2n-1} \end{array}$$

we obtain

$$\begin{array}{rcl} r\times n\times W_{-2n} & = & n\times W_{-2n+1}-s\times n\times W_{-2n-1} \\ r\times(n-1)\times W_{-2n+2} & = & (n-1)\times W_{-2n+3}-s\times(n-1)\times W_{-2n+1} \\ r\times(n-2)\times W_{-2n+4} & = & (n-2)\times W_{-2n+5}-s\times(n-2)\times W_{-2n+3} \\ r\times(n-3)\times W_{-2n+6} & = & (n-3)\times W_{-2n+7}-s\times(n-3)\times W_{-2n+5} \\ & \vdots & \\ r\times 4\times W_{-8} & = & 4\times W_{-7}-s\times 4\times W_{-9} \\ r\times 3\times W_{-6} & = & 3\times W_{-5}-s\times 3\times W_{-7} \\ r\times 2\times W_{-4} & = & 2\times W_{-3}-s\times 2\times W_{-5} \\ r\times 1\times W_{-2} & = & 1\times W_{-1}-s\times 1\times W_{-3} \\ r\times 0\times W_{0} & = & 0\times W_{1}-s\times 0\times W_{-1} \\ r\times(-1)\times W_{2} & = & (-1)\times W_{3}-t\times(-1)\times W_{1} \end{array}$$

If we add the equations by side by, we get

$$r\sum_{i=1}^{n} iW_{-2i} = (\sum_{i=1}^{n} iW_{-2i+1}) - s(nW_{-2n-1} + \sum_{i=1}^{n}(i-1)W_{-2i+1})$$

for $n \geq 1$. Since

$$\sum_{i=1}^{n}(i-1)W_{-2i+1} = \sum_{i=1}^{n} iW_{-2i+1} - \sum_{i=1}^{n} W_{-2i+1}$$

it follows that

$$r\sum_{i=1}^{n} iW_{-2i} = -snW_{-2n-1} + (1-s)\sum_{i=1}^{n} iW_{-2i+1} + s\sum_{i=1}^{n} W_{-2i+1} \tag{3.2}$$

for $n \geq 1$. Then, using Theorem 1.2 (b) and (c) and solving system (3.1)-(3.2) the required result of (b) and (c) follow.

Taking $r = s = 1$ in Theorem 3.1 (a), (b) and (c) we obtain the following proposition.

Proposition 3.1. *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{i=1}^{n} iW_{-i} = -(2n+5)W_{-n-1} - (n+3)W_{-n-2} + 2W_1 + W_0$.

(b) $\sum_{i=1}^{n} iW_{-2i} = -W_{-2n} - nW_{-2n-1} + W_0$.

(c) $\sum_{i=1}^{n} iW_{-2i+1} = -(n+1)W_{-2n} - W_{-2n-1} + W_1$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.2. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{i=1}^{n} iF_{-i} = -(2n+5)F_{-n-1} - (n+3)F_{-n-2} + 2$.

(b) $\sum_{i=1}^{n} iF_{-2i} = -F_{-2n} - nF_{-2n-1}$.

(c) $\sum_{i=1}^{n} iF_{-2i+1} = -(n+1)F_{-2n} - F_{-2n-1} + 1$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.3. *For $n \geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{i=1}^{n} iL_{-i} = -(2n+5)L_{-n-1} - (n+3)L_{-n-2} + 4$.

(b) $\sum_{i=1}^{n} iL_{-2i} = -L_{-2n} - nL_{-2n-1} + 2$.

(c) $\sum_{i=1}^{n} iL_{-2i+1} = -(n+1)L_{-2n} - L_{-2n-1} + 1$.

Taking $r = 2, s = 1$ in Theorem 3.1 (a), (b) and (c) we obtain the following proposition.

Proposition 3.2. *If $r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{i=1}^{n} iW_{-i} = \frac{1}{2}(-(5+3n)W_{-n-1} - (2+n)W_{-n-2} + W_1)$.

(b) $\sum_{i=1}^{n} iW_{-2i} = \frac{1}{4}(-W_{-2n} - 2nW_{-2n-1} + W_0)$.

(c) $\sum_{i=1}^{n} iW_{-2i+1} = \frac{1}{4}(-2(1+n)W_{-2n} - W_{-2n-1} + W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 3.4. *For $n \geq 1$, Pell numbers have the following properties.*

(a) $\sum_{i=1}^{n} iP_{-i} = \frac{1}{2}(-(5+3n)P_{-n-1} - (2+n)P_{-n-2} + 1)$.
(b) $\sum_{i=1}^{n} iP_{-2i} = \frac{1}{4}(-P_{-2n} - 2nP_{-2n-1})$.
(c) $\sum_{i=1}^{n} iP_{-2i+1} = \frac{1}{4}(-2(1+n)P_{-2n} - P_{-2n-1} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.5. *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a) $\sum_{i=1}^{n} iQ_{-i} = \frac{1}{2}(-(5+3n)Q_{-n-1} - (2+n)Q_{-n-2} + 2)$.
(b) $\sum_{i=1}^{n} iQ_{-2i} = \frac{1}{4}(-Q_{-2n} - 2nQ_{-2n-1} + 2)$.
(c) $\sum_{i=1}^{n} iQ_{-2i+1} = \frac{1}{4}(-2(1+n)Q_{-2n} - Q_{-2n-1} + 2)$.

If $r = 1, s = 2$ then $(r-s+1)(r+s-1) = 0$ so we can't use Theorem 3.1 (b) and (c), directly. However, we can find $\sum_{i=1}^{n} iW_{-2i}$ and $\sum_{i=1}^{n} iW_{-2i+1}$ using mathematical induction which is given in the following theorem.

Theorem 3.6. *If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{i=1}^{n} iW_{-i} = \frac{1}{4}(-(6n+11)W_{-n-1} - 2(2n+5)W_{-n-2} + 3W_1 + 2W_0)$.
(b) $\sum_{i=1}^{n} iW_{-2i} = \frac{1}{54}(-(8+3n)W_{-2n} - 2(16+15n)W_{-2n-1} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)n^2)$.
(c) $\sum_{i=1}^{n} iW_{-2i+1} = \frac{1}{54}(-(16+33n)W_{-2n} - 2(32+3n)W_{-2n-1} + 16(2W_1 - W_0) + 9(W_1 - 2W_0)n^2)$.

Proof. (b) and (c) can be proved by mathematical induction.

(a) Taking $r = 1, s = 2$ in Theorem 3.1 (a) we obtain (a).

(b) The proof will be by induction on n. If $n = 1$ we see that the sum formula reduces to the relation

$$W_{-2} = \frac{1}{54}(-(8+3)W_{-2} - 2(16+15)W_{-3} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)). \qquad (3.3)$$

Since

$$\begin{aligned} W_{-2} &= (\frac{3}{4}W_0 - \frac{1}{4}W_1) \\ W_{-3} &= (-\frac{5}{8}W_0 + \frac{3}{8}W_1) \end{aligned}$$

(3.3) is true. Assume that the relation in (b) is true for $n = m$, i.e.,

$$\sum_{i=1}^{m} iW_{-2i} = \frac{1}{54}(-(8+3m)W_{-2m} - 2(16+15m)W_{-2m-1} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)m^2).$$

Then we get

$$\begin{aligned} \sum_{i=1}^{m+1} iW_{-2i} &= (m+1)W_{-2(m+1)} + \sum_{i=1}^{m} iW_{-2i} \\ &= (m+1)W_{-2m-2} + \frac{1}{54}(-(8+3m)W_{-2m} - 2(16+15m)W_{-2m-1} \\ &\quad +8(2W_1 - W_0) - 9(W_1 - 2W_0)m^2) \\ &= \frac{1}{54}(-(8+3m)W_{-2m} - 2(16+15m)W_{-2m-1} + 54(m+1)W_{-2m-2} \\ &\quad +9(W_1 - 2W_0)(2m+1) + 8(2W_1 - W_0) - 9(W_1 - 2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(8+3(m+1))W_{-2m-2} - 2(16+15(m+1))W_{-2m-3} \\ &\quad +8(2W_1 - W_0) - 9(W_1 - 2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(8+3(m+1))W_{-2(m+1)} - 2(16+15(m+1))W_{-2(m+1)-1} \\ &\quad +8(2W_1 - W_0) - 9(W_1 - 2W_0)(m+1)^2) \end{aligned}$$

where

$$\begin{aligned} & -(8+3m)W_{-2m}-2(16+15m)W_{-2m-1}+54(m+1)W_{-2m-2}+9\left(W_1-2W_0\right)(2m+1) \\ = & -(8+3(m+1))W_{-2m-2}-2(16+15(m+1))W_{-2m-3}. \end{aligned} \tag{3.4}$$

(3.4) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n=m+1$.

(c) We now prove (c) by induction on n. If $n=1$ we see that the sum formula reduces to the relation

$$W_{-1}=\frac{1}{54}(-(16+33\times 1)W_{-2}-2(32+3\times 1)W_{-3}+16(2W_1-W_0)+9(W_1-2W_0)\times 1^2) \tag{3.5}$$

Since

$$\begin{aligned} W_{-1} &= (-\frac{1}{2}W_0+\frac{1}{2}W_1) \\ W_{-2} &= (\frac{3}{4}W_0-\frac{1}{4}W_1) \\ W_{-3} &= (-\frac{5}{8}W_0+\frac{3}{8}W_1) \end{aligned}$$

(3.5) is true. Assume that the relation in (c) is true for $n=m$ i,e.,

$$\sum_{i=1}^{m} iW_{-2i+1}=\frac{1}{54}(-(16+33m)W_{-2m}-2(32+3m)W_{-2m-1}+16(2W_1-W_0)+9(W_1-2W_0)m^2).$$

Then we get

$$\begin{aligned} \sum_{i=1}^{m+1} iW_{-2i+1} &= (m+1)W_{-2m-1}+\sum_{i=1}^{m} iW_{-2i+1} \\ &= (m+1)W_{-2m-1}+\frac{1}{54}(-(16+33m)W_{-2m}-2(32+3m)W_{-2m-1} \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)m^2) \\ &= \frac{1}{54}(-(16+33m)W_{-2m}+2\left(24m-5\right)W_{-2m-1}-9\left(2m+1\right)\left(W_1-2W_0\right) \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(16+33(m+1))W_{-2m-2}-2(32+3(m+1))W_{-2m-3}+16(2W_1-W_0) \\ &\quad +9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(16+33(m+1))W_{-2(m+1)}-2(32+3(m+1))W_{-2(m+1)-1}+16(2W_1-W_0) \\ &\quad +9(W_1-2W_0)(m+1)^2) \end{aligned}$$

where

$$\begin{aligned} & -(16+33m)W_{-2m}+2\left(24m-5\right)W_{-2m-1}-9\left(2m+1\right)\left(W_1-2W_0\right) \\ = & -(16+33(m+1))W_{-2m-2}-2(32+3(m+1))W_{-2m-3}. \end{aligned} \tag{3.6}$$

(3.6) can be proved by using Binet formula of W_n. Hence, the relation in (c) holds also for $n=m+1$.

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 3.7. *For $n\geq 1$, Jacobsthal numbers have the following property:*

(a) $\sum_{i=1}^{n} iJ_{-i} = \frac{1}{4}(-(6n+11)J_{-n-1} - 2(2n+5)J_{-n-2} + 3).$

(b) $\sum_{i=1}^{n} iJ_{-2i} = \frac{1}{54}(-(8+3n)J_{-2n} - 2(16+15n)J_{-2n-1} + 16 - 9n^2).$

(c) $\sum_{i=1}^{n} iJ_{-2i+1} = \frac{1}{54}(-(16+33n)J_{-2n} - 2(32+3n)J_{-2n-1} + 32 + 9n^2).$

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.8. *For $n \geq 1,$ Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{i=1}^{n} ij_{-i} = \frac{1}{4}(-(6n+11)j_{-n-1} - 2(2n+5)j_{-n-2} + 7).$

(b) $\sum_{i=1}^{n} ij_{-2i} = \frac{1}{54}(-(8+3n)j_{-2n} - 2(16+15n)j_{-2n-1} + 27n^2).$

(c) $\sum_{i=1}^{n} ij_{-2i+1} = \frac{1}{54}(-(16+33n)j_{-2n} - 2(32+3n)j_{-2n-1} - 27n^2).$

4 CONCLUSION

In this work, a number of sum identities were discovered and proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the special cases of the generalized Fibonacci sequences. All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

COMPETING INTERESTS

Authors has declared that no competing interests exist.

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