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On oscillatory second-order nonlinear delay differential equations of neutral type

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Abstract: In this paper, new sufficient conditions are obtained for oscillation of second-order neutral delay differential equations of the form $\frac{d}{dt}\left[r(t)\frac{d}{dt}\left[x(t)+p(t)x(t-\tau)\right]\right]+q(t)G\big(x(t-\sigma_1)\big)+v(t)H\big(x(t-\sigma_2)\big)=0,\ \ t\geq t_0$, under the assumptions $\int_0^\infty \frac{d\eta}{r(\eta)}=\infty$ and $\int_0^\infty \frac{d\eta}{r(\eta)}<\infty$ for $|p(t)|<+\infty$. Two illustrative examples are included.

Keywords: Oscillation, nonoscillation, nonlinear, delay argument, second-order neutral differential equation.

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1. Introduction

This article is concerned with sufficient conditions for oscillation of a nonlinear neutral second-order delay differential equation

$$\frac{d}{dt}\big[r(t)\frac{d}{dt}z(t)\big]+q(t)G\big(x(t-\sigma_1)\big)+v(t)H\big(x(t-\sigma_2)\big)=0,\ \ t\geq t_0, \tag{1}$$

where $z(t)=x(t)+p(t)x(t-\tau)$ and $p\in PC([t_0,\infty),\mathbb{R})$. We also suppose that the following assumptions hold:

(A_1) $r,q,v\in C([t_0,\infty),[0,\infty))$, $\tau,\sigma_1,\sigma_2\in\mathbb{R}_+$ and $\rho=\max\{\tau,\sigma_1,\sigma_2\}$;
(A_2) $G,H\in C(\mathbb{R},\mathbb{R})$ with $uH(u)>0$ and $yG(y)>0$ for $u,y\neq 0$;
(A_3) $\int_0^\infty \frac{d\eta}{r(\eta)}=\infty$;
(A_4) $\int_0^\infty \frac{d\eta}{r(\eta)}<\infty$.

Baculikova *et al.* [1] have considered the second order delay differential equation of the form

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\left[x(t)+p(t)x(\tau(t))\right]\right]+q(t)x(\sigma(t))+v(t)x(\eta(t))=0, \tag{2}$$

where $r(t),q(t),v(t)\in C([t_0,\infty))$, $r(t),p(t),\tau(t),\sigma(t),\eta(t)\in C^1([t_0,\infty))$ and established several sufficient conditions for oscillation of solution of (2) for $0\leq p(t)<\infty$. Li *et al.* [2] obtained sufficient conditions for oscillation of solution of second order nonlinear neutral differential equations of the form

$$\frac{d}{dt}\left[r(t)\left[\frac{d}{dt}\left[x(t)+p(t)x(t-\tau)\right]\right]^\gamma\right]+q(t)f\big(x(t),x(\sigma(t))\big)=0,$$

where $p,q,r\in C([t_0,+\infty),(0,+\infty))$ and $\gamma\geq 1$ is the quotient of two odd positive integers. In [3], Santra has consider first-order nonlinear neutral delay differential equations of the form

$$\frac{d}{dt}\big[x(t)+p(t)x(t-\tau)\big]+q(t)H\big(x(t-\sigma)\big)=f(t) \tag{3}$$

and
$$\frac{d}{dt}[x(t)+p(t)x(t-\tau)]+q(t)H\big(x(t-\sigma)\big)=0 \tag{4}$$
and studied oscillatory behaviour of the solutions of Equation (3) and Equation (4), under various ranges of $p(t)$. Also, sufficient conditions are obtained for existence of bounded positive solutions of (3). Tripathy *et al.* [4] have established several sufficient conditions for the oscillation of solution of the second order nonlinear neutral delay differential equations of the form
$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\left[x(t)+p(t)x(\tau(t))\right]\right]+q(t)f\big(x(\sigma(t))\big)=0$$
and
$$\frac{d}{dt}\left[r(t)\left[\frac{d}{dt}\left[x(t)+p(t)x(\tau(t))\right]\right]^{\gamma}\right]+q(t)x^{\beta}(\sigma(t))=0,$$
where $r,q,\tau,\sigma\in C(\mathbb{R}_+,\mathbb{R}_+)$, $p\in C(\mathbb{R}_+,\mathbb{R})$ and γ,β are quotient of odd positive integers. Motivated by the above work, an attempt is made to study oscillatory behaviour of Equation (1) for $|p(t)|<+\infty$. Here we are connected with both (A_3) and (A_4).

Neutral functional differential equations have numerous applications in several field of the science as, for example, models of population growth and theory of population dynamics, fractal theory, nonlinear oscillation of earthquake, diffusion in porous media, fractional biological neurons, traffic flow, polymer theology, neural network modeling, fluid dynamics, viscoelastic panel in super sonic gas flow, real system characterized by power laws, electrodynamics of complex medium, sandwich system identification, nuclear reactors mathematical modeling of the diffusion of discrete particles in a turbulent fluid (see [5–7,9] and the references cited therein). In last decades several results have been obtained on oscillation of nonneutral differential equations and neutral functional differential equations (see [10–15] and the references cited therein).

By a solution to Equation (1), we mean a function $x\in C([T_x,\infty),\mathbb{R})$, $T_x\geq t_0$, which has the property $rz'\in C^1([T_x,\infty),\mathbb{R})$ and satisfies Equation (1) on the interval $[T_x,\infty)$. We consider only those solutions to Equation (1) which satisfy condition $\sup\{|x(t)|: t\geq T\}>0$ for all $T\geq T_x$ and assume that Equation (1) possesses such solutions. A solution of Equation (1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be nonoscillatory. Equation (1) itself is said to be oscillatory if all of its solutions are oscillatory.

2. Sufficient Conditions for Oscillation

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of second order nonlinear neutral differential equations of the form (1).

Theorem 1. *Let $0\leq p(t)\leq p<1$, $t\in\mathbb{R}_+$. Assume that (A_1)–(A_3) hold. Furthermore assume that*

(A_5) *G and H are nondecreasing and odd function*

and

(A_6) $\int_T^\infty[q(\eta)+Lv(\eta)]d\eta=\infty$, $L=\frac{H(\varepsilon)}{G(\varepsilon)}>0$ *for* $\varepsilon,T>0$

hold. Then every solution of the equation (1) is oscillatory.

Proof. Suppose for contrary that $x(t)$ is a nonoscillatory solution of equation (1). Then there exists $t_0\geq\rho$ such that $x(t)>0$ or <0 for $t\geq t_0$. Assume that $x(t)>0$, $x(t-\tau)>0$ and $x(t-\sigma)>0$ for $t\geq t_0$. From Equation (1), it follows that
$$\big[r(t)z'(t)\big]'=-q(t)G\big(x(t-\sigma_1)\big)-v(t)H\big(x(t-\sigma_2)\big)<0, \tag{5}$$
hold for $t\geq t_1>t_0$. Consequently, $r(t)z'(t)$ is nonincreasing and $z'(t)$, $z(t)$ are of constant sign on $[t_2,\infty)$ for $t_2>t_1$. Let $r(t)z'(t)<0$ for $t\geq t_2$. Then we can find $\varepsilon_1>0$ and a $t_3>t_2$ such that $r(t)z'(t)\leq-\varepsilon_1$ for $t\geq t_3$. Integrating the relation $z'(t)\leq-\frac{\varepsilon_1}{r(t)}$ from t_3 to $t(>t_3)$ and obtain $z(t)\leq z(t_3)-\varepsilon_1\left[\int_{t_3}^t\frac{d\eta}{r(\eta)}\right]\to-\infty$ as $t\to\infty$, a contradiction to the fact that $z(t)>0$ for $t\geq t_1$. Hence, $r(t)z'(t)>0$ for $t\geq t_2$. As a result, $z(t)$ is

nondecreasing on $[t_2,\infty)$. So, there exists $\varepsilon_2>0$ and a $t_3>t_2$ such that $z(t)\geq\varepsilon_2$ for $t\geq t_3$. On the other hand, $z(t)$ is nondecreasing implies that

$$\begin{aligned}(1-p(t))z(t)&\leq z(t)-p(t)z(t-\tau)\\&=x(t)+p(t)x(t-\tau)-p(t)x(t-\tau)-p(t)p(t-\tau)x(t-2\tau)\\&=x(t)-p(t)p(t-\tau)x(t-2\tau)\leq x(t),\end{aligned}$$

that is, $(1-p)\varepsilon_2\leq x(t)$. Consequently, $x(t)\geq\varepsilon$ where $(1-p)\varepsilon_2=\varepsilon>0$. Therefore, (5) can be written as

$$\big(r(t)z'(t)\big)'+G(\varepsilon)[q(t)+Lv(t)]\leq 0.$$

We note that $\lim_{t\to\infty}r(t)z'(t)$ exists. Integrating the last inequality from t_3 to $t(>t_3)$, then

$$G(\varepsilon)\int_{t_3}^{t}[q(\eta)+Lv(\eta)]d\eta\leq-[r(\eta)z'(\eta)]_{t_3}^{t}<\infty,\ \ as\ \ t\to\infty,$$

a contradiction due to the assumption (A_6).

If $x(t)<0$ for $t\geq t_0$, then we set $y(t)=-x(t)$ for $t\geq t_0$ in (1) and using (A_5) we find

$$\big(r(t)(y(t)+p(t)y(t-\tau))'\big)'+q(t)G\big(y(t-\sigma_1)\big)+v(t)H\big(y(t-\sigma_2)\big)=0,$$

then proceeding as above, we find a same contradiction. This completes the proof of the theorem. □

Theorem 2. *Let $1\leq p(t)\leq p<\infty$, $t\in\mathbb{R}_+$ and $G(p)\geq H(p)$. Assume that (A_1)–(A_3) and (A_5) hold. Furthermore assume that there exists $\lambda,\mu>0$ such that*

(A_7) $G(u)+G(s)\geq\lambda G(u+s)$, $H(u)+H(s)\geq\mu H(u+s)$ *for* $u,s\in\mathbb{R}_+$,
(A_8) $G(us)\leq G(u)G(s)$, $H(us)\leq H(u)H(s)$ *for* $u,s\in\mathbb{R}_+$

and

(A_9) $\int_T^\infty[Q(\eta)+L_1V(\eta)]d\eta=\infty$, $L_1=\frac{\mu H(\varepsilon)}{\lambda G(\varepsilon)}>0$ *for* $T,\varepsilon>0$

hold, where $Q(t)=\min\{q(t),q(t-\tau)\}$, $V(t)=\min\{v(t),v(t-\tau)\}$. Then conclusion of the Theorem 1 is true.

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1). Proceeding as in Theorem 1, we have two cases: $r(t)z'(t)<0$ and $r(t)z'(t)>0$ for $t\in[t_2,\infty)$. The former case follows from Theorem 1. Let's consider the later case. As a result, $z(t)$ is nondecreasing on $[t_2,\infty)$. So, there exists $\varepsilon>0$ and a $t_3>t_2$ such that $z(t)\geq\varepsilon$ for $t\geq t_3$. We note that $\lim_{t\to\infty}r(t)z'(t)$ exists. From Equation (1), it is easy to see that

$$\begin{aligned}0\ &=\ \big(r(t)z'(t)\big)'+q(t)G\big(x(t-\sigma_1)\big)+v(t)H\big(x(t-\sigma_2)\big)+G(p)\big[\big(r(t-\tau)z'(t-\tau)\big)'\\&\quad+q(t-\tau)G\big(x(t-\tau-\sigma_i)\big)+v(t-\tau)H\big(x(t-\tau-\sigma_2)\big)\big],\end{aligned}$$

in which we use (A_7), (A_8) and $z(t)\leq x(t)+px(t-\tau)$ to obtain

$$\begin{aligned}0\ &\geq\ \big(r(t)z'(t)\big)'+G(p)\big(r(t-\tau)z'(t-\tau)\big)'+Q(t)\big[G(x(t-\sigma_1))+G(px(t-\tau-\sigma_1))\big]\\&\quad+v(t)H\big(x(t-\sigma_2)\big)+G(p)v(t-\tau)H\big(x(t-\tau-\sigma_2)\big)\\&\geq\ \big(r(t)z'(t)\big)'+G(p)\big(r(t-\tau)z'(t-\tau)\big)'+\lambda Q(t)G\big[x(t-\sigma_1)+px(t-\tau-\sigma_1)\big]\\&\quad+v(t)H\big(x(t-\sigma_2)\big)+G(p)v(t-\tau)H\big(x(t-\tau-\sigma_2)\big)\\&\geq\ \big(r(t)z'(t)\big)'+G(p)\big(r(t-\tau)z'(t-\tau)\big)'+\lambda Q(t)G\big(z(t-\sigma_1)\big)+v(t)H\big(x(t-\sigma_2)\big)\\&\quad+H(p)v(t-\tau)H\big(x(t-\tau-\sigma_2)\big),\end{aligned}$$

that is,

$$\big(r(t)z'(t)\big)'+G(p)\big(r(t-\tau)z'(t-\tau)\big)'+\lambda Q(t)G\big(z(t-\sigma_1)\big)+\mu V(t)H\big(z(t-\sigma_2)\big)\leq 0\tag{6}$$

for $t\geq t_3>t_2$. Consequently,

$$\big(r(t)z'(t)\big)'+G(p)\big(r(t-\tau)z'(t-\tau)\big)'+\lambda Q(t)G(\varepsilon)+\mu V(t)H(\varepsilon)\leq 0.$$

Integrating the last inequality from t_3 to $t(> t_3)$, then

$$\lambda G(\varepsilon)\int_{t_3}^{t}[Q(\eta)+L_1V(\eta)]d\eta \le -\big[r(\eta)z'(\eta)\big]_{t_3}^{t}+G(p)\big[r(\eta-\tau)z'(\eta-\tau)\big]_{t_3}^{t}<\infty,\ \ as\ \ t\to\infty,$$

a contradiction due to the assumption (A_9). The case $x(t)<0$ is similar. Thus the theorem is proved. □

Theorem 3. *Let $-1\le p(t)\le 0$, $t\in\mathbb{R}_+$. If (A_1)–(A_3), (A_5) and (A_6) hold, then every unbounded solution of Equation (1) oscillates.*

Proof. Let on the contrary that $x(t)$ be a unbounded solution of Equation (1) on $[t_0,\infty)$, $t_0>\rho$. Proceeding as in Theorem 1, it concludes that $r(t)z'(t)$ is nonincreasing and $z(t)$, $z'(t)$ are monotonicon $[t_2,\infty)$. Indeed, $z(t)<0$ for $t\ge t_3$ implies that $x(t)\le x(t-\tau)$, and hence

$$x(t)\le x(t-\tau)\le x(t-2\tau)\le ...\le x(t_3),$$

that is, $x(t)$ is bounded, which is absurd. Hence, $z(t)>0$ for $t\ge t_3$. Suppose that $r(t)z'(t)>0$ for $t\ge t_3$. Clearly, $z(t)\le x(t)$ implies that

$$\big(r(t)z'(t)\big)'+q(t)G\big(z(t-\sigma_1)\big)+v(t)H\big(z(t-\sigma_2)\big)\le 0 \tag{7}$$

for $t\ge t_3$. On the other hand, $z(t)$ is nondecreasing implies that, there exist $\varepsilon>0$ and a $t_4>t_3$ such that $z(t)\ge\varepsilon$ for $t\ge t_4$. Consequently, for $t_5>t_4+\sigma$, it follows from Equation (7) that

$$\big(r(t)z'(t)\big)'+G(\varepsilon)q(t)+H(\varepsilon)v(t)\le 0,\ t\ge t_5$$

Integrating the last inequality from t_5 to t $(>t_5)$, we have

$$G(\varepsilon)\int_{t_5}^{t}[q(\eta)+Lv(\eta)]d\eta\le -\big[r(s)z'(s)\big]_{t_5}^{t}<\infty,\ as\ t\to\infty,$$

a contradiction to (A_6). Hence, $r(t)z'(t)<0$ for $t\ge t_3$. Rest of the theorem follows from Theorem 1. Thus, the proof of the theorem is complete. □

Theorem 4. *Let $-1<-p\le p(t)\le 0$, $t\in\mathbb{R}_+$ and $p>0$. If all the assumptions of Theorem 3 hold, then every solution of Equation (1) either oscillates or converges to zero as $t\to\infty$.*

Proof. Proceeding as in the proof of Theorem 1, we have obtained Equation (5) and hence $r(t)z'(t)$ is nonincreasing on $[t_2,\infty)$. Therefore, $z(t)$ is monotonic on $[t_3,\infty)$, $t_3>t_2$. So we have four cases namely:

1. $z(t)>0,\quad r(t)z'(t)>0,$
2. $z(t)>0,\quad r(t)z'(t)<0,$
3. $z(t)<0,\quad r(t)z'(t)>0,$
4. $z(t)<0,\quad r(t)z'(t)<0.$

Using the arguments as in the proof of Theorems 1 and Theorem 3, we get contradictions to (A_3) and (A_6) when the **Case (2)** and **Case (1)** respectively. Since $z(t)<0$ implies that $x(t)$ is bounded, that is, $z(t)$ is bounded, then the **Case (4)** is not possible due to Theorem 1 ($\because z'(t)<0$ implies that $\lim_{t\to\infty}z(t)=-\infty$).
Consequently, the **Case (3)** holds for $t\ge t_3$. In this case, $\lim_{t\to\infty}z(t)$ exits. As a result,

$$\begin{aligned}
0 &\ge \lim_{t\to\infty}z(t)=\limsup_{t\to\infty}z(t)=\limsup_{t\to\infty}\big(x(t)+p(t)\ x(t-\tau)\big)\\
&\ge \limsup_{t\to\infty}\big(x(t)-p\ x(t-\tau)\big)\\
&\ge \limsup_{t\to\infty}x(t)+\liminf_{t\to\infty}\big(-px(t-\tau)\big)=(1-p)\limsup_{t\to\infty}x(t)
\end{aligned}$$

implies that $\limsup_{t\to\infty}x(t)=0$ $[\because 1-p>0]$ and hence $\liminf_{t\to\infty}x(t)=0$. Thus $\lim_{t\to\infty}x(t)=0$. The case $x(t)<0$ is similar dealt with. This completes the proof of the theorem. □

Theorem 5. *Let $-\infty < -p_1 \le p(t) \le -p_2 < -1$, $p_1, p_2 > 0$ and $t \in \mathbb{R}_+$. Assume that (A_1)–(A_3), (A_5) and (A_6) hold. If*

(A_{10}) $\int_T^\infty [q(\eta) + L_2 v(\eta)]d\eta = \infty$, $L_2 = \frac{H(-p_1^{-1}\alpha)}{G(-p_1^{-1}\alpha)} > 0$ *for* $T, p_1 > 0$ *and* $\alpha < 0$,

then every bounded solution of Equation (1) either oscillates or converges to zero as $t \to \infty$.

Proof. Suppose on the contrary that $x(t)$ is a solution of Equation (1) which is bounded on $[t_0, \infty)$, $t_0 > \rho$. Using the same type of reasoning as in Theorem 1, we have that $z'(t)$ and $z(t)$ are of one sign on $[t_2, \infty)$ and have four possible cases like as in Theorem 4. **Case (2)** and **Case (4)** are not possible because of (A_3) and bounded $z(t)$. **Case (1)** follows from the proof of the Theorem 3. For the **Case (3)**, we claim that $\lim_{t\to\infty} z(t) = 0$. If not, there exists $\alpha < 0$ and $t_3 > t_2$ such that $z(t + \tau - \sigma_1) < \alpha$ and $z(t + \tau - \sigma_2) < \alpha$ for $t \ge t_3$. Hence, $z(t) \ge p(t)x(t - \tau) \ge -p_1 x(t - \tau)$ implies that $x(t - \sigma_1) \ge -p_1^{-1}\alpha > 0$ and $x(t - \sigma_2) \ge -p_1^{-1}\alpha > 0$ for $t \ge t_3$. Consequently, Equation (5) becomes

$$\big(r(t)z'(t)\big)' + G(-p_1^{-1}\alpha)q(t) + H(-p_1^{-1}\alpha)v(t) \le 0 \tag{8}$$

for $t \ge t_3$. Integrating the last inequality from t_3 to $t(> t_3)$, we get

$$G(-p_1^{-1}\alpha)\int_{t_3}^{t} [q(\eta) + L_2 v(\eta)]d\eta \le -\big[r(s)z'(s)\big]_{t_3}^{t} < \infty, \text{ as } t \to \infty,$$

a contradiction to (A_{10}). Ultimately, $\lim_{t\to\infty} z(t) = 0$. Hence,

$$\begin{aligned} 0 &= \lim_{t\to\infty} z(t) = \liminf_{t\to\infty} z(t) \\ &\le \liminf_{t\to\infty}\big(x(t) - p_2\, x(t-\tau)\big) \\ &\le \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-p_2\, x(t-\tau)\big) \\ &= (1 - p_2)\limsup_{t\to\infty} x(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ $[\because 1 - p_2 < 0]$. Thus, $\liminf_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$. Therefore, any solution $x(t)$ of Equation (1) converges to zero. The case $x(t) < 0$ is similar. This completes the proof of the theorem. □

Remark 1. If we denote $R(t) = \int_t^\infty \frac{d\eta}{r(\eta)}$, then (A_4) implies that $R(t) \to 0$ as $t \to \infty$, since $R(t)$ is nonincreasing.

Theorem 6. *Let $0 \le p(t) \le p < \infty$, $t \in \mathbb{R}_+$ and $G(p) \ge H(p)$. Assume that (A_1), (A_2), (A_4), (A_5) and (A_7)–(A_9) hold. If*

(A_{11}) $\int_T^\infty \frac{1}{r(\eta)}\left[\int_{T_1}^{\eta}\{Q(\zeta)G\big(\varepsilon R(\zeta - \sigma_1)\big) + L_3 V(\zeta)H\big(\varepsilon R(\zeta - \sigma_2)\big)\}d\zeta\right]d\eta = \infty$ *for* $T, T_1, C > 0$,

where $L_3 = \frac{\mu}{\lambda} > 0$ then also conclusion of the Theorem 1 is true, where $Q(t)$ and $V(t)$ is defined in Theorem 2.

Proof. On the contrary, we proceed as in Theorem 1 to obtain Equation (5) for $t \ge t_1$ and $r(t)z'(t)$ is non increasing on $[t_2, \infty)$, $t_2 > t_1$. The case $r(t)z'(t) > 0$ for $t \ge t_0$ is same as in Theorem 2 and gives a contradiction due to (A_9). Let's suppose that $r(t)z'(t) < 0$, for $t \ge t_2$. Therefore, for $s \ge t > t_2$, $r(s)z'(s) \le r(t)z'(t)$ implies that

$$z'(s) \le \frac{r(t)z'(t)}{r(s)}.$$

Consequently,

$$z(s) \le z(t) + r(t)z'(t)\int_t^s \frac{d\theta}{r(\theta)}.$$

Because of $r(t)z'(t)$ is nonincreasing, we can find a constant $\varepsilon > 0$ such that $r(t)z'(t) \leq -\varepsilon$ for $t \geq t_2$. As a result, $z(s) \leq z(t) - \varepsilon \int_t^s \frac{d\eta}{r(\eta)}$ and hence $0 \leq z(t) - \varepsilon R(t)$ for $t \geq t_2$. Using the above fact in Equation (6), we get

$$\big(r(t)z'(t)\big)' + G(p)\big(r(t-\tau)z'(t-\tau)\big)' + \lambda Q(t)G\big(\varepsilon R(t-\sigma_1)\big) + \mu V(t)H\big(\varepsilon R(t-\sigma_2)\big) \leq 0$$

for $t \geq t_3 > t_2$. Integrating the last inequality from t_3 to $t(> t_3)$, we obtain

$$\big[r(\eta)z'(\eta)\big]_{t_3}^t + G(p)\big[r(\eta-\tau)z'(\eta-\tau)\big]_{t_3}^t + \lambda \int_{t_3}^t \big[Q(\eta)G\big(\varepsilon R(\eta-\sigma_1)\big) + L_3 V(\eta)H\big(\varepsilon R(\eta-\sigma_2)\big)\big]d\eta \leq 0,$$

that is,

$$\begin{aligned} \lambda \int_{t_3}^t \big[Q(\eta)G(\varepsilon R(\eta-\sigma_1)) + L_3 V(\eta)H\big(\varepsilon R(\eta-\sigma_2)\big)\big]d\eta &\leq -\big[r(\eta)z'(\eta) + G(p)\big(r(\eta-\tau)z'(\eta-\tau)\big)\big]_{t_3}^t \\ &\leq -\big[r(t)z'(t) + G(p)\big(r(t-\tau)z'(t-\tau)\big)\big] \\ &\leq -\big(1+G(p)\big)r(t)z'(t) \end{aligned}$$

implies that

$$\frac{\lambda}{1+G(p)}\frac{1}{r(t)}\int_{t_3}^t \big[Q(\eta)G\big(\varepsilon R(\eta-\sigma_1)\big) + L_3 V(\eta)H\big(\varepsilon R(\eta-\sigma_2)\big)\big]d\eta \leq -z'(t).$$

Again integrating the last inequality, we obtain that

$$\frac{\lambda}{1+G(p)}\int_{t_3}^t \frac{1}{r(\eta)}\left[\int_{t_3}^\eta \{Q(\zeta)G\big(\varepsilon R(\zeta-\sigma_1)\big) + L_3 V(\zeta)H\big(R(\zeta-\sigma_2)\big)\}d\zeta\right]d\eta \leq -\big[z(\eta)\big]_{t_3}^t.$$

Since $z(t)$ is bounded and monotonic, then it follows that

$$\int_{t_3}^t \frac{1}{r(\eta)}\left[\int_{t_3}^\eta \{Q(\zeta)G\big(\varepsilon R(\zeta-\sigma_1)\big) + L_3 V(\zeta)H\big(\varepsilon R(\zeta-\sigma_2)\big)\}d\zeta\right]d\eta < \infty,$$

a contradiction to (A_{11}). The case $x(t) < 0$ is similar dealt with. This completes the proof of the theorem. □

Theorem 7. *Let $-1 \leq p(t) \leq 0$, $t \in \mathbb{R}_+$. Assume that (A_1), (A_2) and (A_4)–(A_6) hold. Furthermore assume that*

(A_{12}) $\int_T^\infty \frac{1}{r(\eta)}\left[\int_{T_1}^\eta \{q(\zeta)G\big(\varepsilon R(\zeta-\sigma_1)\big) + v(\zeta)H\big(\varepsilon R(\zeta-\sigma_2)\big)\}d\zeta\right]d\eta = \infty$ *for* $T, T_1, C > 0$

hold. Then conclusion of the Theorem 3 is true.

Proof. The proof of the theorem follows from the proof of the Theorems 3 and 6 and hence the details are omitted. □

Theorem 8. *Let $-1 < -p \leq p(t) \leq 0$, $t \in \mathbb{R}_+$ and $p > 0$. If all the conditions of Theorem 7 are satisfied, then conclusion of the Theorem 4 is true.*

Proof. The proof of the theorem follows from the proof of Theorems 4 and 7. Hence, the proof of the theorem is complete. □

Theorem 9. *Let $-\infty < -p_1 \leq p(t) \leq -p_2 < -1$, $t \in \mathbb{R}_+$ and $p_1, p_2 > 0$. Assume that (A_1), (A_2), (A_4)–(A_6), (A_{10}) and (A_{12}) hold. If*

(A_{13}) $\int_T^\infty \frac{1}{r(\eta)}\left[\int_{T_1}^\eta \{q(\zeta) + L_2 v(\zeta)\}d\zeta\right]d\eta = \infty$ *for* $T, T_1 > 0$,

where L_2 is defined in Theorem 5, then conclusion of the Theorem 5 is true.

Proof. Proceeding as in the proof of the Theorem 5 we have four possible cases for $t \geq t_2$. First two cases are similar to the proof of Theorem 8. **Case (3)** is similar to the proof of Theorem 5. Hence, we consider the **Case (4)** only. Using the same type of reasoning as in the **Case (3)** of Theorem 8, we get Equation (8) and hence

$$H(-p_1^{-1}\alpha)\left[\int_{t_3}^{t}\{q(\eta)+L_2v(\eta)\}d\eta\right]\leq -r(t)z'(t).$$

Therefore,

$$H(-p_1^{-1}\alpha)\int_{t_3}^{t}\frac{1}{r(\eta)}\left[\int_{t_3}^{\eta}\{q(\zeta)+L_2v(\zeta)\}d\zeta\right]d\eta\leq -\left[z(\eta)\right]_{t_3}^{t}\leq -z(t)<\infty,\ \ as\ \ u\to\infty,$$

a contradiction to (A_{13}). Rest of the theorem follows from the proof of the Theorem 5. This completes the proof of the theorem. □

3. Final Comment and Examples

In this section, we will be giving some simple remarks to conclude the paper.

Remark 2. In Theorem 1–Theorem 9, G and H is allowed to be linear, sublinear or superlinear. A prototype of the function G and H satisfying (A_2), (A_5), (A_7) and (A_8) is

$$(1+\alpha|u|^{\beta})|u|^{\gamma}\mathrm{sgn}(u)\quad \text{for } u\in\mathbb{R}, \tag{9}$$

where $\alpha\geq 1$ or $\alpha=0$ and $\beta,\gamma>0$ are reals. For verifying (A_6), we may take help of the well-known inequality (see [16, p. 292])

$$u^p+v^p\geq h(p)(u+v)^p\quad \text{for } u,v>0,\quad \text{where}\quad h(p):=\begin{cases}1, & 0\leq p\leq 1,\\ \dfrac{1}{2^{p-1}}, & p\geq 1.\end{cases}$$

We finalize the paper by presenting two examples, which show existence of main results.

Example 1. Consider the differential equation

$$\frac{d}{dt}\left[e^{-4t}\frac{d}{dt}\left[x(t)+x(t-\pi)\right]\right]+e^t\big(x(t-\tfrac{\pi}{2})\big)^3+e^t\big(x(t-\tfrac{3\pi}{2})\big)^3=0\quad \text{for } t\geq\pi, \tag{10}$$

where $r(t):=e^{-4t}$, $p(t):\equiv 1$, $\tau:=\pi$, $q(t):\equiv e^t$, $\sigma_1:=\frac{\pi}{2}$, $G(u):=u^3$, $v(t):=e^t$, $\sigma_2=\frac{3\pi}{2}$ and $H(u):=u^3$ for $t\geq\pi$ and $u\in\mathbb{R}$. All the assumptions of Theorem 1 can be verified. Hence, due to Theorem 1, every solution of Equation (10) oscillates. Clearly $x(t)=\sin(t)$ for $t\geq\pi$ is a solution Equation (10).

Example 2. Consider the differential equation

$$\frac{d}{dt}\left[\frac{1}{t^2}\frac{d}{dt}\left[x(t)-e^{-\pi}x(t-\pi)\right]\right]+4\cosh(\pi)t\big[e^{-\frac{\pi}{2}}(t+1)x(t-\tfrac{\pi}{2})+x(t-\pi)\big]=0\quad \text{for } t\geq 2\pi, \tag{11}$$

where $r(t):=\frac{1}{t^2}$, $R(t):=\frac{1}{t}$, $p(t):\equiv e^{-\pi}$, $\tau:=\pi$, $q(t):=4e^{-\frac{\pi}{2}}\cosh(\pi)t(t+1)$, $\sigma_1=\frac{\pi}{2}$, $G(u):=u$, $v(t):=4\cosh(\pi)t$, $\sigma_2:=\pi$ and $H(u):=u$ for $t\geq 2\pi$ and $u\in\mathbb{R}$. All the assumptions of Theorem 7 can be verified. In particular, for $(A11)$, we have

$$\int_{2\pi}^{\infty}\frac{1}{\eta}\int_{2\pi}^{\eta}4\cosh(\pi)\zeta\frac{\varepsilon}{\zeta-\pi}d\zeta d\eta=\infty\quad \text{for any } \varepsilon>0.$$

Hence, due to Theorem 7, every solution of Equation (11) oscillates, and such a solution is $x(t)=e^t\sin(t)$ for $t\geq 2\pi$.

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