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On smarandachely adjacent vertex total coloring of subcubic graphs

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Abstract: Inspired by the observation that adjacent vertices need possess their own characteristics in terms of total coloring, we study the smarandachely adjacent vertex total coloring (abbreviated as SAVTC) of a graph G, which is a proper total coloring of G such that for every vertex u and its every neighbor v, the color-set of u contains a color not in the color-set of v, where the color-set of a vertex is the set of colors appearing at the vertex or its incident edges. The minimum number of colors required for an SAVTC is denoted by $\chi_{sat}(G)$. Compared with total coloring, SAVTC would be more likely to be developed for potential applications in practice. For any graph G, it is clear that $\chi_{sat}(G) \geq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. We, in this work, analyze this parameter for general subcubic graphs. We prove that $\chi_{sat}(G) \leq 6$ for every subcubic graph G. Especially, if G is an outerplanar or claw-free subcubic graph, then $\chi_{sat}(G) = 5$.

Keywords: Smarandachely adjacent vertex total coloring, subcubic graphs, outerplane graphs, claw-free.

MSC: 05C15.

1. Introduction

All graphs considered in this paper are simple and undirected. The terminology and notation used but undefined here can be found in [1]. Let G be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. We use $d_G(v)$ to denote the *degree* of a vertex v in G. A vertex v is called a *t-vertex* (*t^--vertex* or *t^+-vertex*) of G if $d_G(v)$=t ($d_G(v) \leq t$ or $d_G(v) \geq t$). We refer to t-vertices, t^--vertices and t^+-vertices adjacent to v as *t-neighbors*, *t^--neighbors* and *t^+-neighbors* of v, respectively. Let $\Delta(G)$ and $\delta(G)$ denote the *maximum degree* and *minimum degree* of G, respectively. The *open neighborhood* of v, written as $N_G(v)$, is defined as the set of vertices adjacent to v in G, i.e. $N_G(v) = \{u|uv \in E(G)\}$. For any $V' \subset V(G)$ and $E' \in E(G)$, we use $G - V'$ (resp. $G - E'$) to denote the graph obtained from G by deleting vertices in V' and their incident edges (resp. by removing edges in E'). For any integers a, b with $a < b$, let $[a,b]$=$\{a, a+1, \ldots, b\}$.

We, for convenience, denote by $T(G)$ the set of vertices and edges of a graph G, i.e. $T(G) = V(G) \cup E(G)$. Let k be a positive integer, and f a mapping from $T(G)$ to $[1,k]$. If f satisfies the following coloring conditions:

(1) $f(u) \neq f(v)$ for any $uv \in E(G)$,
(2) $f(u) \neq f(e)$ for every vertex u and every edge e incident with u,
(3) $f(e) \neq f(e')$ for every pair e, e' of adjacent edges,

then we call f a *proper total k-coloring* of G. For any $v \in V(G)$, we call $C_f(v)$ the *color-set of v (under f)*, which denotes the set of colors of v and its incident edges under f. Furthermore, let $\overline{C}_f(x) = [1,k] \setminus C_f(x)$. To distinguish the color-sets of two adjacent vertices from the perspective of proper total coloring, Zhang et al.[2] introduced the concept of adjacent vertex distinguishing total coloring (or AVDTC simply), which is a proper total coloring f with the constraint (4) as follows:

(4) $C_f(u) \neq C_f(v)$ for every $uv \in E(G)$.

The minimum number k such that G has a k-AVDTC is the *adjacent vertex distinguishing total chromatic number of G*, denoted by $\chi_{at}(G)$. As for this parameter, a famous conjecture says that every graph G has an adjacent vertex distinguishing total coloring using at most $\Delta(G)+3$ colors, i.e. $\chi_{at}(G) \leq \Delta(G)+3$. This conjecture has been confirmed for special families of graphs, e.g. graphs with maximum degree 3 [3–5], graphs without K_4-minor [6], graphs with smaller maximum average degree and large maximum degree [7,8], outerplane graphs [9], 2-degenerate graphs [10], graphs with maximum degree 4 [11], generalized Mycielski graphs [12], etc. In [13], a stronger version of AVDTC called *smarandachely adjacent vertex total coloring* (abbreviate to SAVTC) is studied. A *k-SAVTC* of a graph G is a proper total k-coloring that satisfies coloring condition (5) as below:

(5) $C_f(u) \setminus C_f(v) \neq \emptyset$ and $C_f(v) \setminus C_f(u) \neq \emptyset$ for every $uv \in E(G)$.

We refer to the smallest number k such that G has a k-SAVTC as the *smarandachely adjacent vertex total chromatic number of* G, denoted by $\chi_{sat}(G)$. Clearly, condition (5) is a stronger version of condition (4). That is, if f is a k-SAVTC of G, then f is a k-AVDTC of G, whereas the converse is not necessarily true. For example, when a graph G contains no adjacent vertices with maximum degree, it is possible that $\chi_{at}(G) = \Delta(G) + 1$, e.g. the star graph $S_n, n \geq 3$. However, by coloring condition (5), one can readily check that $\chi_{sat}(G) \geq \Delta(G) + 2$ for all graphs G.

Therefore, such a parameter is independent, interesting and meaningful. In [13], Zhang proposed the following conjecture.

Conjecture 1. *[13] For any graph G, $\chi_{sat}(G) \leq \Delta(G) + 3$.*

Observe that for two adjacent vertices $u, v \in V(G)$ such that $d_G(u) \leq d_G(v)$, to check that $C_f(u)$ and $C_f(v)$ satisfy the coloring condition (5) under a total coloring f of G, it is sufficient to examine whether there is an element c such that $c \in C_f(u)$ and $c \notin C_f(v)$. Therefore, we have the following lemma, which demonstrates the relation between $\chi_{at}(G)$ and $\chi_{sat}(G)$ for regular graphs G.

Lemma 2. *Let G be a regular graph. Then, $\chi_{at}(G) = \chi_{sat}(G)$.*

To verify our results in this paper, we first introduce a simple but useful lemma as follows.

Lemma 3. *Let A, B be two sets containing p and q elements, respectively. If $p \leq q - 1$ and $A \setminus B \neq \emptyset$, then for any element c, $(A \cup \{c\}) \setminus B \neq \emptyset$ and $B \setminus (A \cup \{c\}) \neq \emptyset$.*

Proof. Since $A \setminus B \neq \emptyset$, there exists an element $a \in A$ and $a \notin B$. In addition, $q \geq p + 1$ implies that B contains at least two distinct elements b_1, b_2 such that $b_1 \notin A$ and $b_2 \notin A$. Therefore, $|B \setminus (A \cup \{c\})| \geq 1$ and $|(A \cup \{c\}) \setminus B| \geq 1$. □

If a graph contains a 1-vertex, then we have the following observation with regard to the SAVTC.

Lemma 4. *Suppose that G is a graph with an 1-vertex u such that $G - \{u\}$ has a k-SAVTC, $k \geq \Delta(G) + 2$. Let $\{v\} = N_G(u)$, $d_G(v) = \ell(\geq 2)$, and N the set of $(\ell - 1)^-$-neighbors of v. If $|N| \leq k - \ell$, then every k-SAVTC f of $G - \{u\}$ can be extended to a k-SAVTC of G.*

Proof. Based on f, edge uv has $k - \ell$ available colors under the coloring conditions (1), (2) and (3). Because $|N| \leq k - \ell$, there exists an available color $\alpha \in [1, k]$ for uv such that $C_f(v') \not\subset (C_f(v) \cup \{\alpha\})$ for any $v' \in N \setminus \{u\}$. By Lemma 3, we have that $(C_f(v) \cup \{\alpha\}) \not\subset C_f(v')$ for any $v' \in N_G(v) \setminus N$. Therefore, we obtain a k-SAVTC of G after coloring u with a color in $[1, k] \setminus (C_f(v) \cup \{\alpha\})$. □

2. Subcubic graphs

A graph G is said to be *cubic* if $\delta(G) = \Delta(G) = 3$ and *subcubic* if $\Delta(G) \leq 3$. Since $\chi_{at}(G) \leq 6$ for every cubic graph G [3–5], it has that $\chi_{sat}(G) \leq 6$ by Lemma 2. In this section, we aim to extend this result from cubic graphs to subcubic graphs. We prove the following theorem.

Theorem 5. *If G is a subcubic graph, then $\chi_{sat}(G) \leq 6$.*

Proof. It is sufficient to deal with the case that G contains a 2^--vertex. Let G be a counterexample to Theorem 5 such that $|E(G)|$ is minimum, and v a 2^--vertex. We will prove that G contains a 6-SAVTC, and get a contradiction. If $d_G(v) = 1$, then by Lemma 4 we can get a 6-SAVTC of G from any 6-SAVTC of $G - \{v\}$.

Therefore, we assume $d_G(v) = 2$, and suppose that G does not contain any 1-vertex. Let $\{u, w\}$ be the open neighborhood of v.

Case 1. At least one of these two vertices, say u, is a 2-vertex. Let $u' = N_G(u) \setminus \{v\}$, and by the minimality f' a 6-SAVTC of $G - \{vu\}$. We now extend f' by the following rule: if $(C_{f'}(u) \cup C_{f'}(w)) \neq [1,6]$, let $\alpha \in [1,6] \setminus (C_{f'}(u) \cup C_{f'}(w))$, and assign color α to uv and recolor v with a color in $[1,6] \setminus (\{\alpha, f'(vw), f'(w)\} \cup C_{f'}(u))$ (observe that $|C_{f'}(u)| = 2$). Denote by f the resulting coloring. Obviously, $\alpha \notin C_f(w)$, $f(v) \notin C_f(u)$, and by Lemma 3 $C_f(u) \not\subset C_f(u')$. This shows that f is a 6-SAVTC of G; if $(C_{f'}(u) \cup C_{f'}(w)) = [1,6]$, then $C_{f'}(u) \cap C_{f'}(w) = \emptyset$. We therefore obtain a 6-SAVTC of G by coloring uv with $f'(w)$ and recoloring v with $f'(uu')$.

Case 2. Both u and w are 3-vertices. When $uw \in E(G)$, we will extend a 6-SAVTC f' of $G - \{uv\}$ to a such coloring of G. Let $\{u'\} = N_G(u) \setminus \{v, w\}$. Observe that $|C_{f'}(u) \cup \{f'(vw)\}| \leq 4$. We can choose a color $\alpha \in [1,6] \setminus (C_{f'}(u) \cup \{f'(vw)\})$ such that $C_{f'}(u') \not\subset (C_{f'}(u) \cup \{\alpha\})$. By Lemma 3, $(C_{f'}(u) \cup \{\alpha\}) \not\subset C_{f'}(w)$. If $|C_{f'}(u) \cup C_{f'}(w) \cup \{\alpha\}| \neq 6$, then we can recolor v with a color in $[1,6] \setminus (C_{f'}(u) \cup C_{f'}(w) \cup \{\alpha\})$ to get a 6-SAVTC of G. If $|C_{f'}(u) \cup C_{f'}(w) \cup \{\alpha\}| = 6$, then either $\alpha \notin C_{f'}(w)$ and $f'(vw) \notin C_{f'}(u)$, or $\alpha \in C_{f'}(w)$ and $f'(vw) \notin C_{f'}(u)$ (or $\alpha \notin C_{f'}(w)$ and $f'(vw) \in C_{f'}(u)$). In the former case, we can recolor v with a color in $[1,6] \setminus \{\alpha, f'(u), f'(vw), f'(w)\}$ to get a 6-SAVTC of G, while in the latter eventuality we can recolor v with a color in $[1,6] \setminus (C_{f'}(w) \cup \{f'(u)\})$ (or $[1,6] \setminus (C_{f'}(u) \cup \{\alpha, f'(w)\})$) to gain a 6-SAVTC of G. In the remainder of this proof, let $uw \notin E(G)$.

Consider $G' = (G - \{v\}) \cup \{uw\}$. We see that G' is a subcubic graph and $|E(G')| < |E(G)|$. By the minimality, G' has a 6-SAVTC f'. Without loss of generality, we may suppose that $C_{f'}(u) = \{1,2,3,4\}$, where $f'(u) = 4, f'(uw) = 1$. Let $N_G(u) = \{v, u_1, u_2\}$ and $N_G(w) = \{v, w_1, w_2\}$. Suppose that $\overline{C}_{f'}(w) = \{c_1, c_2\}$. We now extend f' to a 6-SAVTC of G by addressing the following two situations.

When $1 \notin \{f'(u_i), f'(w_i) | i = 1,2\}$, we recolor u and w with 1, color uv with 4 and vw with $f'(w)$. Denote by f the resulting coloring. Clearly, $C_f(u) = C_{f'}(u)$ and $C_f(w) = C_{f'}(w)$. If $f'(w) \in \{5,6\}$, then $f'(w) \notin C_f(u)$. Therefore, after coloring v with a color in $\{c_1, c_2\} \setminus \{4\}$, we get a 6-SAVTC of G. If $f'(w) \notin \{5,6\}$, it has that $f'(w) \in \{2,3\}$. Then, when $4 \in \{c_1, c_2\}$, we can color v with 5 or 6 to get a 6-SAVTC of G; when $4 \notin \{c_1, c_2\}$, it follows that $\{5,6\} \cap \{c_1, c_2\} \neq \emptyset$. Therefore, we obtain a 6-SAVTC of G by coloring v with a color in $\{5,6\} \cap \{c_1, c_2\}$.

When $1 \in \{f'(u_i), f'(w_i) | i = 1,2\}$, we, by symmetry, assume that $f'(u_1) = 1$. In this case, we first color vw with 1, and vu with a color $c \in \{5,6\}$ such that $C_{f'}(u_2) \not\subset \{2,3,4,c\}$. Then, color v with a color in $\{c_1, c_2\} \setminus \{4, c\}$ if $\{4, c\} \neq \{c_1, c_2\}$ (observe that $f'(w) \notin \{c_1, c_2\}$); otherwise, color v with a color in $\{2,3\} \setminus \{f'(w)\}$. Denote by f the resulting coloring. It has that $C_f(w) = C_{f'}(w)$, $C_f(u) = \{2,3,4,c\}$, $1 \in C_f(v)$, $1 \in C_f(u_1)$, $1 \notin C_f(u)$, and $f(v) \notin C_f(w)$ or $c \notin C_f(w)$. Therefore, f is a 6-SAVTC of G. □

3. Graphs with smarandachely adjacent vertex total chromatic number 5

In this section, we aim to construct a 5-SAVTC for the given classes of subcubic graphs. For this, we will get a 5-SAVTC of a hypothetical smallest counterexample G to the theorem we need prove by extending a 5-SAVTC f' of a smaller graph derived from G, and obtain a contradiction. In the process of extending f', we by default color elements shared by G and G' with the restriction of f' to them if there is no specified note.

3.1. Outerplanar graphs with maximum degree 3

A planar graph G is called *outerplanar* if there is an embedding of G into the Euclidean plane such that all vertices lie on the boundary of its unbounded face. An outerplanar graph equipped with such an embedding is called an *outerplane graph*. To show that outerplane graphs with maximum degree 3 have a 5-SAVTC, we need the following lemma.

Lemma 6. *Suppose that f is a partial coloring of the graph G shown in Figure 1 (a), where $V(G) = \{v, x, y, x_1, y_1\}$ and $f(x_1) = c_1, f(y_1) = c_2, f(x_1x) = c_3, f(y_1y) = c_4$. If $|\{c_i | i = 1,2,3,4\}| \geq 3$, $c_1 \neq c_2, c_1 \neq c_3, c_2 \neq c_4$ and $c_3 \neq c_4$, then we can construct a 5-SAVTC of G on the restriction of f.*

Proof. In Figures 1 (b) and (c), we give the corresponding 5-SAVTCs of G for the cases $|\{c_i | i = 1,2,3,4\}| = 3$ and $|\{c_i | i = 1,2,3,4\}| = 4$, respectively. Observe that under each 5-SAVTC of G, c_1 and c_2 is not in the color-set of x and y, respectively, and c_1, c_2 belong to the color-set of v. □

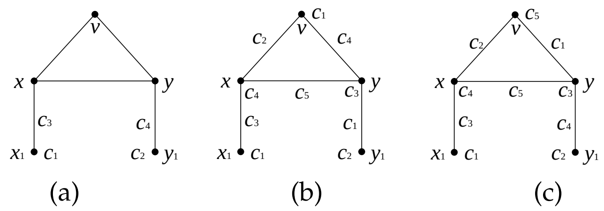


Figure 1. A graph and its certain colorings

Theorem 7. *Let G be an outerplane graph with maximum degree 3. Then, $\chi_{sat}(G) = 5$.*

Proof. It is enough to show that G has a 5-SAVTC. Let G be a counterexample to Theorem 7 with minimum number of edges. We distinguish two cases.

Case 1. G contains a cut-vertex v. Then, there are two smaller outerplane graphs G_1 and G_2 such that $\Delta(G_i) \leq 3, i = 1, 2$, $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{v\}$. By the minimality, G_i has a 5-SAVTC, denoted by $f_i, i = 1, 2$. Without loss of generality, suppose that $d_{G_1}(v) \leq 2$, $d_{G_2}(v) = 1$ and $N_{G_2}(v) = \{u\}$.

Case 1.1. $|V(G_2)| = 2$, i.e. $G_2 = vu$. If $d_{G_1}(v) = 1$, then by Lemma 4 any 5-SAVTC of G_1 can be extended to a 5-SAVTC of G. We therefore assume that $d_{G_1}(v) = 2$ and $N_{G_1}(v) = \{v_1, v_2\}$. If $\{v_1, v_2\}$ contains an 1-vertex or 3-vertex, say v_1, we extend f_1 to a 5-SAVTC f by coloring vu with a color $\alpha \in [1, 5] \setminus C_{f_1}(v)$ such that $C_{f_1}(v_2) \not\subset C_{f_1}(v) \cup \{\alpha\}$ (since $|[1, 5] \setminus C_{f_1}(v)| = 2$, such a color α does exist), and color u (or recolor v_1 when $d_{G_1}(v_1) = 1$) with β, where $\{\beta\} = [1, 5] \setminus (C_{f_1}(v) \cup \{\alpha\})$. Since $\beta \notin C_f(v)$ and by Lemma 3 $C_f(v) \neq C_f(v_1)$ when $d_{G_1}(v_1) = 3$, f is a 5-SAVTC of G.

Now, suppose that $d_{G_1}(v_1) = d_{G_1}(v_2) = 2$ and $N_{G_1}(v_i) = \{v, v'_i\}$, $i = 1, 2$. We omit the trivial case $v_1v_2 \in E(G_1)$ and by Lemma 4 assume $d_{G'}(v'_i) \geq 2$ for $i = 1, 2$. By the minimality, let f' be a 5-SAVTC of $G - \{v\}$. Based on f', if $C_{f'}(v_1) \cap C_{f'}(v_2)$ contains an element γ, then we color v, vu, vv_1, vv_2 with $[1, 5] \setminus \{\gamma\}$ properly and color u with γ; if $C_{f'}(v_1) \cap C_{f'}(v_2) = \emptyset$, we recolor v_1 with a color $\gamma \in (\{f'(v_2), f'(v_2v'_2)\} \setminus \{f'(v'_1)\})$ and color vv_1 with $f'(v_1)$, vv_2 with $f(v_1v'_1)$, v with the color in $[1, 5] \setminus (C_{f'}(v_1) \cup C_{f'}(v_2))$, vu with $\{f'(v_2), f'(v_2v'_2)\} \setminus \{\gamma\}$ and u with γ. Denote by f the resulting coloring. Since $\gamma \notin C_f(v)$ and by Lemma 3 $C_f(v_i) \not\subset C_f(v'_i)$ for $i = 1, 2$, f is a 5-SAVTC of G.

Case 1.2. $|V(G_2)| \geq 3$. Let $G'_1 = G_1 \cup vu$, $G'_2 = G_2$. By the minimality, G'_1 and G'_2 have a 5-SAVTC f'_1 and f'_2, respectively. By the color permutation, we assume $f'_1(v) = f'_2(v) = 1, f'_1(vu) = f'_2(vu) = 2$ and $f'_1(u) = f'_2(u) = 3$. Clearly, $3 \notin C_{f'_1}(v)$ and $1 \notin C_{f'_2}(u)$. Let $f = f'_1 \cup f'_2$. We see that $C_f(v) = C_{f'_1}(v)$, $C_f(u) = C_{f'_2}(u)$ and $3 \in C_f(u), 1 \in C_f(v)$. Therefore, f is a 5-SAVTC of G.

Case 2. G is 2-connected. We claim that G does contain two adjacent 2-vertices. If not, suppose that u, v are such 2-vertices, where $N_G(u) = \{v, u_1\}$ and $N_G(v) = \{u, v_1\}$. Since G is 2-connected, $d_G(u_1) \geq 2$ and $d_G(v_1) \geq 2$. We first consider the case of $u_1v_1 \in E(G)$. In this case, $d_{G'}(u_1) = d_{G'}(u_2) = 3$. Given a 5-SAVTC f' of $G - \{u\}$, for which we suppose that $\overline{C}_{f'}(v_1) = \{5\}$. Obviously, $5 \in C_{f'}(u_1)$ and $f'(v) = 5$. Based on f', color uu_1 with a color $\alpha \in [1, 5] \setminus C_{f'}(u_1)$ such that $C_{f'}(u_2) \not\subset (C_{f'}(u_1) \cup \{\alpha\})$ (since $|C_{f'}(u_1)| = 3$), where $\{u_2\} = N_G(u_1) \setminus \{v_1, u\}$. Let $\{\beta\} = [1, 5] \setminus (C_{f'}(u_1) \cup \{\alpha\})$, and color u with β and uv with a color in $[1, 5] \setminus \{\alpha, \beta, f'(vv_1), 5\}$. Observe that $5 \notin \{\alpha, \beta\}$; we obtain a 5-SAVTC of G. Now, we assume that $u_1v_1 \notin E(G)$. Let $G' = (G - \{u, v\}) \cup \{u_1v_1\}$, and, by the minimality, f' be one of its 5-SAVTCs. Since $u_1v_1 \in E(G')$, there exist $\alpha_1 \in \overline{C}_{f'}(u_1)$ and $\alpha_2 \in \overline{C}_{f'}(v_1)$ such that $\alpha_1 \neq \alpha_2$. Then, f' can be extended to a 5-SAVTC of G by assigning color $f'(u_1v_1)$ to uu_1 and vv_1, color α_1 to u, color α_2 to v, and a color in $[1, 5] \setminus \{f'(u_1v_1), \alpha_1, \alpha_2\}$. This contradicts the assumption of G.

From the foregoing discussion, we deduce that G contains a triangle $uvwu$ such that $d_G(u) = 2$ and $d_G(v) = d_G(w) = 3$. Let $N_G(v) = \{u, w, v'\}$ and $N_G(w) = \{u, v, w'\}$. Clearly, $d_G(v') \geq 2$ and $d_G(w') \geq 2$. Let $G' = (G - \{v, w\}) \cup \{uv', uw'\}$. By the minimality, G' has a 5-SAVTC, say f'. Suppose that $f'(uv') = c_1, f'(uw') = c_2$, $f'(v') = c_3$ and $f'(w') = c_4$, where $c_i \in [1, 5]$ for $i \in [1, 4]$. We now define a partial coloring g of G such that $g(x) = f'(x)$ for any $x \in T(G) \setminus \{u, v, w, uv, uw, vw, vv'ww'\}$, $g(vv') = f'(uv') = c_1$ and $g(ww') = f'(uw') = c_2$. We see that $C_g(y) = C_{f'}(y)$ for any $y \in V(G) \setminus \{u, v, w\}$, and only elements in $\{u, v, w, uv, uw, vw\}$ are uncolored. Observe that $c_1 \neq c_2$, $c_1 \neq c_3$ and $c_2 \neq c_4$. It suffices to deal with the situation of $c_3 = c_4$ (if $c_3 \neq c_4$, then by Lemma 6 g can be extended to a 5-SAVTC of G). Since $d_G(v') \geq 2$, there exists a color $\alpha \in C_{f'}(v')$ such that $\alpha \notin \{c_1, c_3\}$. We color uv with c_3, w with c_1, vw with a color $\beta \in [1, 5] \setminus \{c_1, c_2, c_3, \alpha\}$, v with a color $\gamma \in [1, 5] \setminus \{c_1, c_3, \alpha, \beta\}$, uw with a color $\alpha' \in [1, 5] \setminus \{c_1, c_3, c_2, \beta\}$, and color

u with α (when $\alpha' \neq \alpha$) or with β (when $\alpha' = \alpha$). It is clear from the resulting coloring, say f, that $c_3 \notin C_f(w)$, $\alpha \notin C_f(v)$ and $\{\alpha, c_3\} \subset C_f(u)$. Therefore, f is a 5-SAVTC of G. This completes the proof. □

3.2. Claw-free subcubic graphs

A graph is called *claw-free* if it contains no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. We will show that every claw-free subcubic graph has a 5-SAVTC. To see this, we first investigate an interesting class of claw-free subcubic graphs as follows.

We use $\mathcal{D}$ to denote the family of graphs, in which every one is obtained from a cubic graph such that every vertex is incident with exactly one triangle by subdividing all edges not incident with triangles, where to subdivide an edge e is to delete e and add a new vertex v, and join v to the ends of e. By this definition, we see that for every graph $G \in \mathcal{D}$, $\delta(G) = 2$, 2-vertices are independent and not incident with any triangle, and the subgraph induced by 3-vertices are the union of vertex-disjoint triangles. We prove that every such graph has a 5-SAVTC.

Theorem 8. *For any $G \in \mathcal{D}$, $\chi_{sat}(G) = 5$.*

Proof. $\chi_{sat}(G) \geq 5$ is obvious. We will give a construction of a 5-SAVTC of G by applying the famous Hall's theorem on bipartite graphs. Let V_2 and V_3 be the set of 2-vertices and 3-vertices of G, respectively. Then, V_2 is an independent set, and the subgraph induced by V_3 is the union of vertex-disjoint triangles, say $T_1, T_2, \ldots, T_m$. Now, we construct a bipartite graph G' with bipartition (X, Y), where $X = V_2, Y = \{T_1, T_2, \ldots, T_m\}$, and $xT \in E(G')$ for $x \in X$, $T \in Y$, if and only if x is adjacent to a vertices incident with T in G. By the definition, we see that $d_{G'}(x) = 2$ for every $x \in X$ and $d_{G'}(T) = 3$ for every $T \in Y$. Therefore, by the Hall's theorem, G' has a matching M_1 which covers every vertex in Y. Consider $G' - M_1$; we have that $d_{G'-M_1}(x) \leq 2$ for every $x \in X$ and $d_{G'-M_1}(T) = 2$ for every $T \in Y$. Again by the Hall's theorem, $G' - M_1$ has a matching M_2 which covers every vertex in Y.

We assert that $G' - M_1$ contains such a M_2 that covers every 2-vertex in X. If not, select M_2 to be the one that covers the most number of 2-vertices in X. Let x be a 2-vertex in X not covered by M_2. Then, there is an M_2-alternating path P starting with x and end with a 1-vertex in X, and $M_2 \triangle E(P)$ (the symmetric difference of M_2 and $E(P)$) is also a matching of $G' - M_1$ which covers every vertex in Y and covers more number of 2-vertices than M_2, a contradiction.

Let M_1 and M_2 be the two matchings selected as above. Then, in $G' - (M_1 \cup M_2)$, $x \in X$ is a 1^--vertex and $T \in Y$ is a 1-vertex. Now, we present an algorithm to construct a 5-SAVTC f of G as follows:

1: For each T_i, $i \in [1, m]$, we use [1,3] to color its vertices and edges, for which the coloring conditions (1), (2) and (3) are satisfied. That is, let $V(T_i) = \{u_i, v_i, w_i\}$, we color u_i, v_i, w_i with 1,2,3 respectively, and color u_iv_i, v_iw_i, w_iu_i with 3,1,2 respectively.
2: Color each edge in M_1 with 4 and in M_2 with 5.
3: Observe that each $v \in V_2$ is an 1^--vertex in $G - (M_1 \cup M_2)$. For each edge $xy \in E(G) \setminus (M_1 \cup M_2)$ such that $x \in V_2$ and $y \in V_3$, let xy' be another edge incident with x in G and assume xy' is colored by α. Clearly, $\alpha \in [4, 5]$ since $xy' \in M_1 \cup M_2$. Suppose that y and y' are colored with β and β' respectively, $\beta, \beta' \in [1, 3]$. If $\beta \neq \beta'$, we recolor y with α, and color xy with $[4, 5] \setminus \{\alpha\}$ and color x with β; if $\beta = \beta'$, then we recolor the vertices and edges of the triangle $T \in Y$ incident with y such that β is not assigned to y (observe that each triangle T_i has only one vertex incident with uncolored edges in this step. Therefore, each such triangle T is recolored at most once. This implies that the recoloring approach does not destroy the coloring before this action). Thus, we can likewise recolor y with α, and color xy with $[4, 5] \setminus \{\alpha\}$ and x with the color appearing at y.
4: After the above three steps, only some 2-vertices are not colored (if possible). Let x be a such uncolored 2-vertex. Suppose that $N_G(x) = \{x_1, x_2\}$, and let α_1 and α_2 be the colors appearing at x_1 and x_2. We then use the color $[1, 3] \setminus \{\alpha_1, \alpha_2\}$ to color x.

According to the above coloring, we see that for each triangle T_i, $i \in [1, m]$, $\{\overline{C}_f(u_i), \overline{C}_f(v_i), \overline{C}_f(w_i)\} = \{4, 5, \beta\}$ where $\beta \in [1, 3]$; for each 2-vertex x, $C_f(x) = \{4, 5, \beta\}$ and $\{\overline{C}_f(x_1), \overline{C}_f(x_2)\} \subset \{\{4, \beta\}, \{5, \beta\}, \{4, 5\}\}$ where $\{x_1, x_2\} = N_G(x)$. Therefore, f is a 5-SAVTC of G. □

Theorem 9. *Let G be a claw-free subcubic graph. Then, $\chi_{sat}(G) = 5$.*

Proof. Suppose to the contrary that G is a counterexample to Theorem 9 such that $E(G)$ is the minimum. It is sufficient to prove that G has a 5-SAVTC. With a similar proof as that in Theorem 7, we have the following claims.

Claim A. G is 2-connected, and G contains no adjacent 2-vertices and triangles incident with a 2-vertex.

To round off the proof, we have to deal with some reducible configurations.

Claim B. G does not contains configurations $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, as shown in Figure 2 (a), (d) and (g).

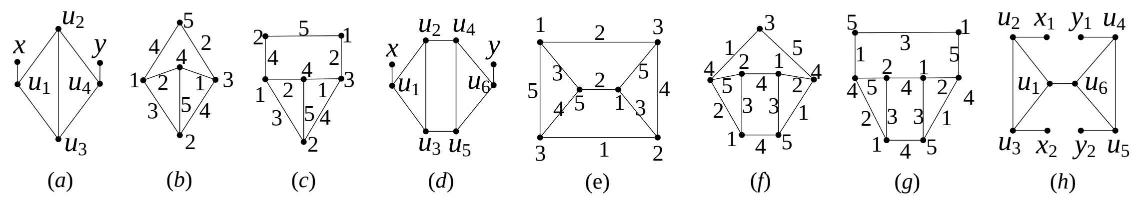


Figure 2. unavoidable configurations

Proof of the claim B. We will show that each of these configurations is reducible, i.e. G has a 5-SAVTC if G contains one of them. Observe that K_4 has a 5-SAVTC. Therefore, we assume that $G \neq K_4$ in what follows.

Case 1. For $\mathcal{H}_1$, since $G \neq K_4$ and G is claw-free, we, by Claim A, may assume that $x \neq u_4, y \neq u_1, x \neq y$ and $xy \neq E(G)$ (if $x = y$ or $xy \in E(G)$, then G is isomorphic to the graph shown in Figure 2 (b) or (c), which has a 5-SAVTC)). Let $G' = (G - \{u_i | i \in [1,4]\}) \cup \{xy\}$. Then, G' is claw-free and subcubic.

By the minimality, G' admits a 5-SAVTC, say g. Without loss of generality, assume $g(x) = 1, g(y) = 2$ and $g(xy) = 3$. We can extend g to a 5-SAVTC of G by coloring elements in $T(G) \setminus T(G')$ as follows: assign color 3 to xu_1, u_2u_3 and yu_4, color 4 to u_1 and u_2u_4, color 5 to u_4 and u_1u_3, color 2 to u_3 and u_1u_2, and color 1 to u_2 and u_3u_4.

Case 2. As for $\mathcal{H}_2$, if $u_1u_6 \in E(G)$, $x = y$ or $xy \in E(G)$, then by Claim A G is isomorphic to the graph shown in Figure 2 (e), (f) or (g), which has a 5-SAVTC. Let $G' = (G - \{u_i | i \in [1,6]\}) \cup \{xy\}$. Then, G' is claw-free and subcubic. By the minimality G' has a 5-SAVTC g. We, without loss of generality, assume that $g(x) = 1, g(y) = 2$ and $g(xy) = 3$. Now, based on the restriction of g to $T(G) \cap T(G')$, we construct f by letting $f(xu_1) = f(yu_6) = f(u_2u_3) = f(u_4u_5) = 3$, $f(u_1) = f(u_4) = f(u_3u_5) = 2$, $f(u_3) = f(u_6) = f(u_2u_4) = 1$, $f(u_1u_2) = f(u_5u_6) = 4$ and $f(u_2) = f(u_5) = f(u_1u_3) = f(u_4u_6) = 5$. Then, $C_f(x) = C_g(x)$, $C_f(y) = C_g(y)$, $\overline{C}_f(u_1) = \overline{C}_f(u_5) = \{1\}$, $\overline{C}_f(u_2) = \overline{C}_f(u_6) = \{2\}$, $\overline{C}_f(u_3) = \overline{C}_f(u_4) = \{4\}$. Hence f is a 5-SAVTC of G.

Case 3. Consider $\mathcal{H}_3$. By Claim A, Case 1 and Case 2, we suppose that $x_1 \neq x_2$, $y_1 \neq y_2$, $x_i \notin \{u_4, u_5\}$, $y_i \notin \{u_2, u_3\}$, $x_1x_2 \notin E(G)$ and $y_1y_2 \notin E(G)$. Let $G' = (G - \{u_i | i \in [1,6]\}) \cup \{x_1x_2, y_1y_2\}$. Obviously, G' is claw-free subcubic graphs with $|E(G')| < |E(G)|$. By the choice of G, G' has a 5-SAVTC g. Without loss of generality, we assume that $g(x_1) = 1, g(x_2) = 2, g(x_1x_2) = 3, g(y_1) = c_1, g(y_2) = c_2$, and $g(y_1y_2) = c_3$, where $c_i \in [1,5]$ for $i = 1,2,3$ and $c_i \neq c_j$ for $1 \leq i < j \leq 3$. We now construct a 5-SAVTC f of G based on the restriction of g to $T(G) \cap T(G')$. We first assign color c_3 to y_1u_4 and y_2u_5, and color 3 to x_1u_2 and x_2u_3. Clearly, $C_f(t) = C_g(t)$ for any $t \in \{x_1, x_2, y_1, y_2\}$.

Case 3.1. $\{c_1, c_2\} \cap \{1,2\} \neq \emptyset$. Then by symmetry we assume that $c_1 = 1$. Let $[1,5] = \{c_1, c_2, c_3, c_4, c_5\}$, and set $f(u_4) = f(u_5u_6) = c_5$, $f(u_5) = f(u_6u_1) = c_1$, $f(u_4u_5) = c_4$, $f(u_4u_6) = c_2$, $f(u_6) = c_3$, $f(u_2) = f(u_3u_1) = 5$, $f(u_2u_3) = 4$, $f(u_2u_1) = 2$, $f(u_3) = 1$, and finally color u_1 with a color in $\{3,4\} \setminus \{c_3\}$ when $c_4 \in \{2,5\}$ or with the color c_4 when $c_4 \in \{3,4\}$. According to the definition of f, we have that $\overline{C}_f(u_2) = \{1\}, \overline{C}_f(u_3) = \{2\}, \overline{C}_f(u_4) = \{c_1\}, \overline{C}_f(u_5) = \{c_2\}, \overline{C}_f(u_6) = \{c_4\}$, $\{1,2,c_4\} \subset C_f(u_1)$. Thus we obtain a 5-SAVTC of G.

Case 3.2. $\{c_1, c_2\} \cap \{1,2\} = \emptyset$. When $c_3 \notin \{1,2\}$, or $c_3 \in \{1,2\}$ and $(\{1,2\} \setminus \{c_3\}) \in C_f(y_i)$ for some $i \in \{1,2\}$, we by symmetry assume that $c_3 = 1$ when $c_3 \in \{1,2\}$, and suppose that $2 \in C_f(y_1)$ (observe that if $c_3 \notin \{1,2\}$, then since $d_G(y_i) \geq 2$ for $i = 1,2$, there exists a color, say 2 here, in $\{1,2\} \cap C_f(y_i)$ for some $i \in \{1,2\}$). Let $\{\alpha\} = \{1,5\} \setminus \{c_1, c_2, c_3, 2\}$, and set $f(u_4) = f(u_5u_6) = \alpha$, $f(u_5) = f(u_6u_1) = 2$, $f(u_4u_5) = c_1$, $f(u_4u_6) = c_2$, $f(u_6) = c_1$, $f(u_3) = f(u_1u_2) = 5$, $f(u_2u_3) = 4$, $f(u_3u_1) = 1$, $f(u_2) = 2$ and color u_1 with c_3 (when $c_3 \notin \{1,5\}$) or a color in $\{3,4\} \setminus \{c_1\}$ (when $c_3 \in \{1,5\}$). Under such coloring f, it follows that $\overline{C}_f(u_2) = \{1\}, \overline{C}_f(u_3) = \{2\}, \overline{C}_f(u_4) = \{2\}, \overline{C}_f(u_5) = \{c_2\}, \overline{C}_f(u_6) = \{c_3\}$ and $\{1,2,c_3\} \subset \overline{C}_f(u_1)$, and hence f is a 5-SAVTC of G.

When $c_3 \in \{1,2\}$ and $(\{1,2\} \setminus \{c_3\}) \notin C_f(y_i)$ for $i = 1,2$, it has that $d_G(y_1) = d_G(y_2) = 2$ (otherwise $C_g(y_1) \subseteq C_g(y_2)$ or $C_g(y_2) \subseteq C_g(y_1)$).

Suppose that $c_3 = 1$ and let $N_G(y_1) = \{u_4, y'\}$. Then, $d_G(y') = 3$ and y' is incident with a triangle. If $g(y') \neq 1$, we recolor y_1 with 1 and color u_4y_1 with c_1. Let $\{\alpha\} = \{1,5\} \setminus \{c_1, c_2, 1, 2\}$, and set $f(u_4) =$

$f(u_5u_6) = 2$, $f(u_5) = c_1$, $f(u_4u_5) = f(u_6) = \alpha$, $f(u_4u_6) = c_2$, $f(u_2) = f(u_3u_1) = 5$, $f(u_2u_3) = 4$, $f(u_2u_1) = 2$, $f(u_3) = f(u_1u_6) = 1$, and finally color u_1 with c_1 (when $c_1 \neq 5$) or a color in $\{3,4\} \setminus \{\alpha\}$ (when $c_1 = 5$). Since $\overline{C}_f(u_2) = \{1\}, \overline{C}_f(u_3) = \{2\}$, $\overline{C}_f(u_4) = \{1\}$, $\overline{C}_f(u_5) = \{c_2\}$, $\overline{C}_f(u_6) = \{c_1\}$ and $\{1,2,c_1\} \subset \overline{C}_f(u_1)$, f is a 5-SAVTC of G.

If $g(y') = 1$, then $c_1 \notin C_f(y')$. We recolor y_1 with the color $\beta \in ([1,5] \setminus \{1, c_2, c_1, g(y_1y')\})$, and color u_4y_1 with c_1. Clearly, $2 \in \{\beta, g(y_1y')\}$. Let $\{\alpha'\} = \{\beta, g(y_1y')\} \setminus \{2\}$, and set $f(u_4) = 1$, $f(u_5u_6) = c_1$, $f(u_5) = 2$, $f(u_4u_5) = f(u_6) = \alpha'$, $f(u_4u_6) = c_2$, $f(u_3) = f(u_2u_1) = 5$, $f(u_2u_3) = 4$, $f(u_3u_1) = 1$, $f(u_2) = f(u_1u_6) = 2$, and color u_1 with a color in $\{3,4\} \setminus \{\alpha'\}$. It is easy to see that $\overline{C}_f(u_2) = \{1\}, \overline{C}_f(u_3) = \{2\}$, $\overline{C}_f(u_4) = \{2\}$, $\overline{C}_f(u_5) = \{c_2\}, \overline{C}_f(u_6) = \{1\}$ and $\{1,2\} \subset \overline{C}_f(u_1)$. Therefore, f is a 5-SAVTC of G.

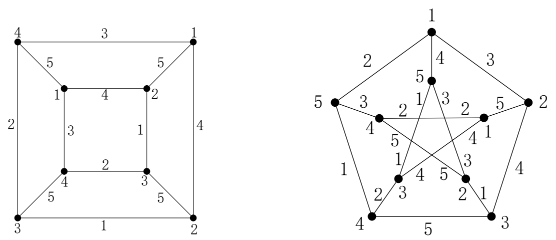
By Claims A and B, we see that G is a 2-connected claw-free subcubic graph which does not contain adjacent 2-vertices, triangles incident with 2-vertices, two triangles sharing a common edge or connecting by an edge (i.e. an edge whose ends incident with two distinct triangles). This indicates that $G \in \mathcal{D}$, and by Theorem 8 G has a 5-SAVTC. This completes the proof of the theorem. □

4. Remarks

For two graphs G and H, let $\sigma : V(G) \to V(H)$ be a surjection. If for every $v \in V(G)$, the restriction of σ to the open neighbourhood of v in G is a bijection onto the open neighbourhood of $\sigma(v)$ in H, i.e. $\sigma(N_G(v)) = N_H(\sigma(v))$, then we call σ a *covering map* from G to H. If there exists a covering map from G to H, then G is called a *covering graph* of H. As for covering graphs, we have the following conclusion on SAVTC.

Theorem 10. *Let H be a graph containing a k-SAVTC g. Then, every covering graph G of H has a k-SAVTC.*

Proof. Let σ be a covering map from G to H. We now use σ to lift a proper total k-coloring, denoted by f, of G, i.e. let $f(v) = g(\sigma(v))$ for every $v \in V(G)$ and $f(uw) = g(\sigma(u)\sigma(w))$ for every $uw \in E(G)$. According to the definition of covering map, if $uw \in E(G)$ then $\sigma(u)\sigma(w) \in E(H)$. We have $f(u)(= g(\sigma(u))) \neq f(w)(= g(\sigma(w)))$ for every $uw \in E(G)$, $f(vu)(= g(\sigma(v)\sigma(u))) \neq f(vw)(= g(\sigma(v)\sigma(w)))$ for any $vu, vw \in E(G)$, and $f(v)(= g(\sigma(v))) \neq f(vu)(= g(\sigma(v)\sigma(u)))$ for any $v \in V(G)$ and $vu \in E(G)$. This shows that f is a proper total k-coloring of G. Moreover, it is easy to see that $C_f(v) = C_g(\sigma(v))$ for every $v \in V(G)$. Therefore, f is a k-SAVTC of G. □



(a) cube hexahedron (b) Petersen graph

Figure 3. Two graphs with 5-SAVTC

In this paper, we discuss an interesting graph parameter χ_{sat}, called the smarandachely adjacent vertex total chromatic number. We derive upper bound for subcubic graphs G, i.e. $\chi_{sat}(G) \leq 6$. We show, in particular, that if G is an outerplane graph with maximum degree 3 or a claw-free subcubic graph, then $\chi_{sat}(G) = 5$. There are also other classes of subcubic graphs with a 5-SAVTC, e.g., the subcubic bipartite graphs. Indeed, for any bipartite G with bipartition (X,Y), we can easily give a $(\Delta(G)+2)$-SAVTC by assigning color 1 to vertices in X, color 2 to vertices in Y, and coloring $E(G)$ by $[3, \Delta(G)+2]$ (since the edge chromatic number of bipartite graphs is the maximum degree). Additionally, by Theorem 10, we can also address a series of subcubic graphs with a 5-SAVTC, which are non-outplanar and contain a claw, for example, the covering graphs of cube hexahedron or Petersen graph; see Figure 3.

In consideration of our conclusions, we propose the following problem:

Problem 11. Let G be a subcubic graph. Is it true that $\chi_{sat}(G) \leq 5$?

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References

[1] Bondy, J., & Murty, U. (1976). *Graph theory with applications*. North-Holland, New York.
[2] Zhang, Z., Chen, X., Li, J., Yao, B., Lu, X., & Wang, J. (2005). On the adjacent-vertex-distinguishing total coloring of graphs. *Science China Mathematics Series A 48*, 289–299.
[3] Chen, X. (2008). On the adjacent vertex distinguishing total coloring numbers of graphs with $\Delta = 3$. *Discrete Mathematics 308*, 4003–4007.
[4] Hulgan, H. (2009). Concise proofs for adjacent vertex-distinguishing total colorings. *Discrete Mathematics 309*, 2548–2550.
[5] Wang, H. (2007). On the adjacent vertex distinguishing total chromatic numbers of graphs with $\Delta = 3$. *Journal of Combinatorial Optimization 14*, 87–109.
[6] Wang, W., & Wang, P. (2009). Adjacent vertex distinguising total coloring of k_4-minor free graphs. *Sci China Ser A 39*, 1462–1472.
[7] Huang, D., & Wang, W. (2012). Adjacent vertex distinguising total coloring of planar grpahs with large maximum degree. *Sci Sin Math 42*, 151–164.
[8] Wang, W., & Wang, Y. (2008). Adjacent vertex distinguising total coloring of planar grpahs with lower average degree. *Taiwanese Journal of Mathematics 12*, 979–990.
[9] Wang, W., & Wang, P. (2010). Adjacent vertex distinguising total colorings of outerplanar graphs. *Journal of Combinatorial Optimization 19*, 123–133.
[10] Miao, Z., Shi, R., Hu, X., & Luo, R. (2016). Adjacent-vertex-distinguishing total coloring of 2-degenerate graphs. *Discrete Mathematics 339*, 2446–2449.
[11] Lu, Y., Li, J., Luo, R., & Miao, Z. (2017). Adjacent vertex distinguishing total coloring of graphs with maximum degree 4. *Discrete Mathematics 340*, 119–123.
[12] Zhu, E., Liu, C., & Xu, J. (2017). On adjacent vertex-distinguishing total chromatic number of generalized mycielski graphs. *Taiwanese Journal of Mathematics 21(2)*, 253–266.
[13] Zhang, Z. (2009). Smarandachely adjacent vertex total coloring of graphs. *The Scientific report of Lanzhou Jiaotong University*, 2–3.