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Global Analysis of Solutions of a New Class of Rational Difference Equation

O. Moaaz[1], E. M. Elabbasy[1], Sh. Alsaeed[2]

[1]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt
[2]Department of Mathematics, The Faculty of Education, Al al-Bayt University, Mafraq, Jordan
Email: emelabbasy@mans.edu.eg, o_moaaz@mans.edu.eg, shaimaa.alsaeed@yahoo.com

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Abstract

The study suggests asymptotic behavior of the solution to a new class of difference equations: $\psi_{\eta+1} = a + \sum_{i=0}^{k} b_i \frac{\psi_{\eta-(2i+1)}}{\alpha\psi_{\eta-2i} + \beta\psi_{\eta-(2i+1)}}$, $\eta = 0,1,2,\cdots$. where a, b_i, α and β are positive real numbers for $i = 0,1,\cdots,k$, and the initial conditions $\psi_{-j}, \psi_{-j+1}, \cdots, \psi_0$ are randomly positive real numbers where $j = 2k+1$. Accordingly, we consider the stability, boundedness and periodicity of the solutions of this recursive sequence. Indeed, we give some interesting counter examples in order to verify our strong results.

Keywords

Difference Equation, Stability, Boundedness, Globale Stability and Periodicity

1. Introduction

The global asymptotic behavior of the solutions and oscillation of solution are two such qualitative properties which are very important for applications in many areas such as control theory, mathematical biology, neural networks, etc. It is impossible to use computer based (numerical) techniques to study the oscillation or the asymptotic behavior of all solutions of a given equation due to the global nature of these properties. Therefore, these properties have received the attention of several mathematicians and engineers.

Currently, much attention has given to study the properties of the solutions of the recursive sequences from scientists in various disciplines. Specifically, the topics dealt with include the following:

- Finding equilibrium points for the recursive sequences;
- Investigating the local stability of the solutions of the recursive sequences;

- Finding conditions which insure that the solutions of the recursive sequences are bounded;
- Investigating the global asymptotic stability of the solutions of the recursive sequences;
- Finding conditions which insure that the solutions of equation are periodic with positive prime period two or more;
- Finding conditions for oscillation of solutions.

Closely related global convergence results were well-gained from these articles [1]-[25]. Khuong in [14] studied the dynamics the recursive sequences

$$\psi_{\eta+1} = a + \left(\frac{\psi_{\eta-k}}{\psi_{\eta-r}} \right)^p .$$

For further related and special cases of this difference equations see [4] [5] [6], [21] [22] [24].

Elsayed [9] studied the periodicity, the boundedness of the positive solution of the recursive sequences

$$\psi_{\eta+1} = a + \frac{b\psi_{\eta-l} + c\psi_{\eta-k}}{c\psi_{\eta-l} + d\psi_{\eta-k}}.$$

Abdelrahman [1] considered analytical investigation of the solution of the recursive sequence

$$\psi_{\eta+1} = a\psi_{\eta-k} + \delta \left(\frac{b\psi_{\eta-l}}{c\psi_{\eta-k} + d\psi_{\eta-l}} \right)^\alpha .$$

By new method, Elsayed [10] investigated the periodic solution of the equation

$$\psi_{\eta+1} = \alpha + \beta \frac{\psi_\eta}{\psi_{\eta-1}} + \gamma \frac{\psi_{\eta-1}}{\psi_\eta}.$$

Also, Moaaz [18] completed the results of [10].

In this work, we deal with some qualitative behaviour of the solutions of the recursive sequence

$$\psi_{\eta+1} = a + \sum_{i=0}^{k} b_i \frac{\psi_{\eta-(2i+1)}}{\alpha\psi_{\eta-2i} + \beta\psi_{\eta-(2i+1)}}, \quad \eta = 0,1,2,\cdots \tag{1.1}$$

where a, b_i, α and β are positive real numbers for $i = 0,1,\cdots,k$, and the the initial conditions $\psi_{-j}, \psi_{-j+1}, \cdots, \psi_0$ are arbitrary positive real numbers where $j = 2k+1$.

In the next, we will and to many of the basic concepts. Before anything, the concept of equilibrium point is essential in the study of the dynamics of any physical system. A point $\bar{\psi}$ in the domain of the function Φ is called an **equilibrium point** of the equation

$$\psi_{\eta+1} = \Phi\left(\psi_\eta, \psi_{\eta-1}, \cdots, \psi_{\eta-k}\right), \quad \eta = 0,1,2,\cdots \tag{1.2}$$

if $\bar{\psi}$ is a fixed point of Φ [$\Phi\left(\bar{\psi}, \bar{\psi}, \cdots, \bar{\psi}\right) = \bar{\psi}$]. For a stability of equilibrium point, equilibrium point $\overline{\psi}$ of equation (2) is said to be **locally stable** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\psi_{-\nu} \in (0,\infty)$ for $\nu = 0,1,\cdots,k$ with $\sum_{i=0}^{k} |\psi_{-i} - \overline{\psi}| < \delta$. As well, $\overline{\psi}$ is said to be **locally asymptotically stable** if it is locally stable and there exists $\gamma > 0$ such that, if $\psi_{-\nu} \in (0,\infty)$ for $\nu = 0,1,\cdots,k$ with $\sum_{i=0}^{k} |\psi_{-i} - \overline{\psi}| < \gamma$, then $\lim_{\eta\to\infty} \psi_\eta = \overline{\psi}$. Also, $\overline{\psi}$ is said to be a **global attractor** if used for every $\psi_{-\nu} \in (0,\infty)$ for $\nu = 0,1,\cdots,k$, we have $\lim_{\eta\to\infty} \psi_\eta = \overline{\psi}$. On the other hand, $\overline{\psi}$ is said to be unstable if it is not locally stable.

Finally, Equation (1.2) is called **permanent and bounded** if there exists numbers r and R with $0 < r < R < \infty$ such that for any initial conditions $\psi_{-\nu} \in (0,\infty)$ for $\nu = 0,1,\cdots,k$ there exists a positive integer N which depends on these initial conditions such as $r < \psi_\eta < R$ for all $\eta \ge N$.

The linearized equation of Equation (1.1) about the equilibrium point $\hat{\psi}$ is

$$y_{\eta+1} = \sum_{i=0}^{k} p_i y_{\eta-i} \tag{1.3}$$

where

$$p_i = \frac{\partial F}{\partial \psi_{\eta-i}} (\hat{\psi}, \hat{\psi}, \cdots, \hat{\psi}).$$

Theorem 1.1. [15] *Assume that* $p_i \in \mathbb{R}$ *for* $i = 0,1,\cdots,k$. *The* $+ve$ *equilibrium of* (1.1) *is locally asymptotically stable if*

$$|p_0| + |p_1| + \cdots + |p_k| < 1. \tag{1.4}$$

2. Local Stability of Equation (1.1)

The $+ve$ equilibrium point of Equation (1.1) is

$$\hat{\psi} = a + \sum_{i=0}^{k} \frac{b_i \hat{\psi}}{\alpha \hat{\psi} + \beta \hat{\psi}},$$

and so,

$$\hat{\psi} = a + \frac{B}{\alpha + \beta},$$

where

$$B = \sum_{i=0}^{k} b_i.$$

Let $f \in C\left((0,\infty)^{2k+2}, (0,\infty)\right)$ defined by

$$f(u_0, u_1, \cdots, u_{2k+1}) = a + \sum_{i=0}^{k} \frac{b_i u_{2i+1}}{\alpha u_{2i} + \beta u_{2i+1}}. \tag{2.1}$$

Therefore it follows that

$$\frac{\partial f}{\partial u_{2r}} = -\frac{b_r \alpha u_{2r+1}}{(\alpha u_{2r} + \beta u_{2r+1})^2} \tag{2.2}$$

and

$$\frac{\partial f}{\partial u_{2r+1}} = \frac{b_r \alpha u_{2r}}{\left(\alpha u_{2r} + \beta u_{2r+1}\right)^2}, \tag{2.3}$$

for $r = 0,1,\cdots,k$.

Theorem 2.1. *Let* $\hat{\psi}$ *be* +ve *equilibrium of Equation* (1.1). *If*

$$\left(\alpha - \beta\right) B < a\left(\alpha + \beta\right)^2,$$

than $\hat{\psi}$ is locally stable.

Proof. From (2.2) to (2.3), we obtain

$$\frac{\partial f}{\partial u_{2r}}\left(\hat{\psi},\cdots,\hat{\psi}\right) = -\frac{b_r \alpha}{\left(\alpha+\beta\right)\left(a\left(\alpha+\beta\right)+B\right)} = p_{2r},$$

and

$$\frac{\partial f}{\partial u_{2r+1}}\left(\hat{\psi},\cdots,\hat{\psi}\right) = \frac{b_r \alpha}{\left(\alpha+\beta\right)\left(a\left(\alpha+\beta\right)+B\right)} = p_{2r+1},$$

for $r = 0,1,\cdots,k$. Thus, the linearized equation of (1.1) is

$$y_{\eta+1} = p_0 y_\eta + p_1 y_{\eta-1} + \cdots + p_{2k+1} y_{\eta-(2k+1)}.$$

It follows by Theorem 1.1 that Equation (1.1) is locally stable if

$$\left|\frac{b_0 \alpha}{\left(\alpha+\beta\right)}\right| + \left|\frac{b_0 \alpha}{\left(\alpha+\beta\right)\rho}\right| + \cdots + \left|\frac{b_k \alpha}{\left(\alpha+\beta\right)\rho}\right| + \left|\frac{b_k \alpha}{\left(\alpha+\beta\right)\rho}\right| < 1,$$

where $\rho = \left(a\left(\alpha+\beta\right)+B\right)$, and hence,

$$\frac{2\alpha}{\left(\alpha+\beta\right)\rho} B < 1.$$

Thus, we find

$$2\alpha B < \left(\alpha+\beta\right)\rho,$$

and so,

$$\left(\alpha - \beta\right) B < a\left(\alpha+\beta\right)^2.$$

Hence, the proof is complete.

□

In order to verify and support our theoretical outcomes and discussions, in this concern, we investigate several interesting numerical examples.

Example 2.1. *By Theorem* 2.1, *the* +ve *equilibrium* Equation (1.1) *with* $a = 2$, $k = 2$, $b_i = 0.3$, $\beta = 0.1$ *and* $\alpha = 1$, *is locally stable* (*see* Figure 1).

3. Global Stability of Equation (1.1)

In the following theorem, we check into the global stability of the recursive sequence (1.1).

Theorem 3.1. *The* +ve *equilibrium* $\hat{\psi}$ *of* Equation (1.1) *is global attractor if*

$$B = a\left(\alpha - \beta\right).$$

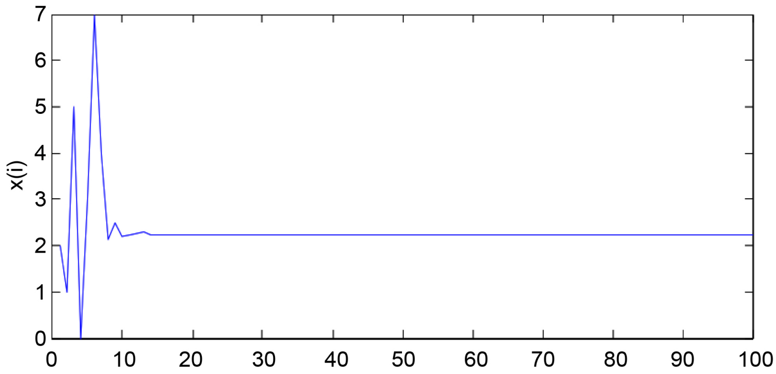

Figure 1. The stable solution corresponding to difference Equation (1.1).

Proof. We consider the function as follow:

$$f\left(u_0,u_1,\cdots,u_{2k+1}\right)=a+\sum_{i=0}^{k}\frac{b_i u_{2i+1}}{\alpha u_{2i}+\beta u_{2i+1}}.$$

From (2.2) and (2.3), we note that f is increasing in u_{2r+1} and decreasing in u_{2r} for all $r=0,1,\cdots,k$. Suppose that (λ,μ) is a solution of the system

$$\lambda=f\left(\mu,\lambda,\mu,\lambda,\cdots,\lambda\right)$$

$$\mu=f\left(\lambda,\mu,\lambda,\mu,\cdots,\mu\right).$$

Then, we find

$$\lambda=a+\sum_{i=0}^{k}b_i\frac{\lambda}{\alpha\mu+\beta\lambda},$$

and

$$\mu=a+\sum_{i=0}^{k}b_i\frac{\mu}{\alpha\lambda+\beta\mu}.$$

Hence, we get

$$\lambda=a+\frac{B\lambda}{\alpha\mu+\beta\lambda}, \tag{3.1}$$

and

$$\mu=a+\frac{B\mu}{\alpha\lambda+\beta\mu}. \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$\beta\left(\lambda^2-\mu^2\right)-\left(a\beta+B-a\alpha\right)\left(\lambda-\mu\right)=0.$$

Thus,

$$\left(\lambda-\mu\right)\left(\beta\left(\lambda+\mu\right)-\left(a\beta+B-a\alpha\right)\right)=0.$$

Since $B=a\left(\alpha-\beta\right)$, we have that $\mu=\lambda$. Hence, the proof of Theorem 3.1 is complete. □

4. Periodic Solutions

In this section, we enumerate some basic facts concerning the existence of two period solutions.

Theorem 4.1. Equation (1.1) *has prime period-two solutions if*

$$(a\beta + B - a\alpha)(\alpha - \beta) > 4a\alpha\beta. \tag{4.1}$$

Proof. Assume that Equation (1.1) has a prime period-two solution

$$\cdots, \rho, \sigma, \rho, \sigma, \rho, \sigma, \cdots$$

We shall prove that condition (4.1) holds. From Equation (1.1), we see that

$$\psi_\eta = \psi_{\eta-2} = \cdots = \psi_{\eta-2k} = \sigma,\ \psi_{\eta+1} = \psi_{\eta-1} = \cdots = \psi_{\eta-(2k+1)} = \rho,$$

and hence,

$$\rho = a + \frac{B\rho}{\alpha\sigma + \beta\rho}, \quad \sigma = a + \frac{B\sigma}{\alpha\rho + \beta\sigma}.$$

Thus, we get

$$\alpha\rho\sigma + \beta\rho^2 = a\alpha\sigma + a\beta\rho + B\rho, \tag{4.2}$$

and

$$\alpha\rho\sigma + \beta\rho^2 = a\alpha\sigma + a\beta\sigma + B\sigma. \tag{4.3}$$

From (4.3) and (4.2), we have

$$\beta(\rho^2 - \sigma^2) = a\alpha(\sigma - \rho) + a\beta(\rho - \sigma) + B(\rho - \sigma).$$

Dividing $(\rho - \sigma)$, then we find

$$(\rho + \sigma) = \frac{a\beta + B - a\alpha}{\beta} \tag{4.4}$$

By combining (4.2) and (4.3), we obtain

$$2\alpha\rho\sigma + \beta(\rho^2 + \sigma^2) = (a\alpha + a\beta + B)(\rho + \sigma).$$

Since $\rho^2 + \sigma^2 = (\rho + \sigma)^2 - 2\rho\sigma$, we get

$$\rho\sigma = \frac{a\alpha(a\beta + B - a\alpha)}{\beta(\alpha - \beta)}. \tag{4.5}$$

Now, evident is that (4.4) and (4.5) that ρ and σ are both two positive distinct roots of the quadratic equation

$$u^2 + (\rho + \sigma)u + \rho\sigma = 0. \tag{4.6}$$

Hence, we obtain

$$\frac{(a\beta + B - a\alpha)}{\beta} > \frac{4a\alpha}{\alpha - \beta},$$

which has the same extent as

$$(a\beta + B - a\alpha)(\alpha - \beta) > 4a\alpha\beta.$$

Hence, the proof is complete. □

The next numerical example is mimicry to enhance our results.

Example 4.1. *By Theorem* 4.1, Equation (1.1) *with* $\alpha = 0.02$, $\beta = 0.01$, $a = 500$, $b_0 = 2$, $b_1 = 200$ *and* $b_2 = 20$, *has prime period two solution* (*see*

Figure 2)

5. Boundedness

Theorem 5.1. *Every solution of* Equation (1.1) *is bounded and persists.*

Proof. Let $\{\psi_\eta\}_{\eta=-k}^{\infty}$ be a Solution (1.1), we can conclude from (1.1) that

$$\psi_{\eta+1} = a + \sum_{i=0}^{k} b_i \frac{\psi_{\eta-(2i+1)}}{\alpha\psi_{\eta-2i} + \beta\psi_{\eta-(2i+1)}} > a.$$

Then

$$\psi_\eta > a \text{ for all } \eta > -(2k+1).$$

Also, from Equation (1.1), we see that

$$\begin{aligned} \psi_{\eta+1} &= a + \sum_{i=0}^{k} b_i \frac{\psi_{\eta-(2i+1)}}{\alpha\psi_{\eta-2i} + \beta\psi_{\eta-(2i+1)}} \\ &< a + \sum_{i=0}^{k} b_i \frac{\psi_{\eta-(2i+1)}}{\beta\psi_{\eta-(2i+1)}} \\ &= a + \sum_{i=0}^{k} \frac{b_i}{\beta}, \end{aligned}$$

then,

$$a < \psi_\eta \le a + \frac{B}{\beta} \text{ for all } \eta > -(2k+1).$$

Thus, the solution is bounded and persists and the proof is complete. □

Conclusion 1. *In this paper, we study a asymptotic behavior of solutions of a general class of difference* Equation (1.1). *Our results extend and generalize to the earlier ones. Moreover, we obtain the next results.*

- The +*ve* equilibrium point $\hat{\psi}$ of Equation (1.1) is local stable if $(\alpha-\beta)B < a(\alpha+\beta)^2$. Also, if $B = a(\alpha-\beta)$, then $\hat{\psi}$ is global attractor.

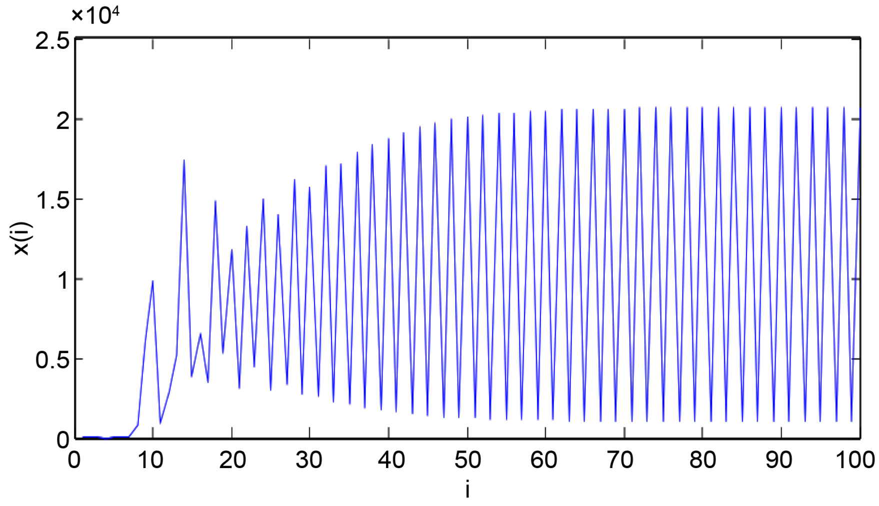

Figure 2. Prime period two solution of Equation (1.1).

- Equation (1.1) has a prime period-two solutions if $(a\beta + B - a\alpha)(\alpha - \beta) > 4a\alpha\beta$.
- Every solution of (1.1) is bounded and persists.

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