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Internal Exact Controllability of the Heat Equation in Finite Dimension by Strategic Actuators Zone

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Authors' contributions

This work was carried out in collaboration among all the authors. Author MTN designed the study and wrote the first draft of the manuscript. The mathematical analysis of the problem was performed by authors AS and CS. Furthermore authors CS and MLLA managed the structural consistency and the english language. While authors CS and MLLA controlled the calculations and the formulas analysis. The three authors managed the literature review, read and approved the final manuscript.

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Abstract

In this paper, we show, by a new approach called SCD (Surjectivity by compactness and density), the internal exact controllability of the 1D heat equation by the use of strategic zone actuators. For the achievement of this objective, an operator (linear, continuous and surjective) built directly from the system allowed us to establish the exact result of controllability.

Keywords: Control; controllability; strategic actuators; estimations.

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1 Motivations and Statement of Problem

The main mathematical models, used in the most diverse applications, are written in the form of partial differential equations. An equation with partial derivatives (PDE) is therefore a relation between a function of several variables and its derivatives. Note that physical models can lead to solutions that are not derivable in the classical sense. So the theory of control and the exact controllability of distributed systems has in recent years a net renewed interest; thanks to the development by J.L Lions [1] of the HUM method (Hilbert Uniqueness Methods). It is essentially based on the uniqueness of properties suitable for the homogeneous equation by a particular choice of controls, building a Hilbert space and a continuous linear mapping of this space into its dual hilbert which is, in fact , an isomorphism which establishes the exact controllability.

For hyperbolic problems this method has yielded important results Lions [1, 2], Seck [3], Lebeau [4], Fursikov [5],...; Although when the checks are supporting small Niane [6], Jai [7], it seems little operative, as well as for technical reasons, the multiplier method does not lead to.

So for parabolic equations any result (to our knowledge) has been prepared by this method. Also, the harmonic method is ineffective for such equations.

This work is the logical sequence of the publications (see Seck [8]) where a new method has been developped to solves some of these problems types. This method is based on criteria of surjectivity of a linear continuous operator a Hilbert space in another builds directly from the exact controllability problem.

The criteria are of two types:

1. A density criterion which is a consequence of the uniqueness properties (J. L. Lions [1, 2]);
2. A compactness criterion which follows from the parabolic nature of the operator or the regularity of control;

In both cases these criteria are easier to verify than HUM method of Lions.

This method which we call **exact controllability by surjective, compactness and density** opens up broad prospects for the theory of the exact controllability. Also allows to for the equations parabolic, of Schrödinger, plates, Navier Stokes linearized to solve many issues and so opening up many perspectives.

2 Notations and Some Functional Analysis Reminders

2.1 Functional analysis reminders

Let Ω be a regular nonempty bounded domain of $\mathbb{R}^N$ of border Γ, of outer unit normal $\nu(\sigma)$ at point $\sigma \in \Gamma$.

We denote A the unbounded operator of $L^2(\Omega)$ defined by

$$D(A) = \left\{u \in H_0^1(\Omega)/ - \triangle u \in L^2(\Omega)\right\} \tag{2.1}$$

$$Au = -\triangle u, \ \forall u \in D(A) \tag{2.2}$$

The operator A has a Hilbert base of eigenfunctions $(w_k)_{k\geq 1}$ such that the sequence of associated eigenvalues $(\lambda_k)_{k\geq 1}$ is decreasing (see Jai [9]).

Let $T > 0$, we consider the following Hilberts spaces and their respective dual.

Let $\alpha \in \mathbb{R}, \delta \in \{1, 0, -1\}$.

$$F_T^{\alpha,\delta} = \left\{ u = \sum_{k=1}^{+\infty} u_k w_k / \sum_{k=1}^{+\infty} \lambda_k^{\alpha} u_k^2 e^{2\lambda_k T\delta} < +\infty \right\} \tag{2.3}$$

We equip $F_T^{\alpha,\delta}$ with the natural scalar product

$$(x, y)_{F_T^{\alpha,\delta}} = \sum_{k=1}^{+\infty} u_k w_k \tag{2.4}$$

and, the associeted norm is $||.||_{F_T^{\alpha,\delta}}$.
The dual of $F_T^{\alpha,\delta}$ is $F_T^{-\alpha,-\delta}$. If $x \in F_T^{\alpha,\delta}$ and $y \in F_T^{-\alpha,-\delta}$ we have:

$$(y, x)_{F_T^{-\alpha,-\delta}, F_T^{\alpha,\delta}} = \sum_{k=1}^{+\infty} x_k y_k \tag{2.5}$$

Remark 2.1. We can notice that:

i. $F_T^{0,0} = L^2(\Omega)$,
ii. $F_T^{1,0} = H_0^1(\Omega)$,
iii. $F_T^{-1,0} = H^{-1}(\Omega)$

2.2 Case of heat equation 1-D

Let $I = \Omega =]0, \pi[$ an open interval of $\mathbb{R}$ and the 1D heat equation defined as follows:

$$\begin{cases} y'(t,x) - \triangle y(t,x) = \beta(t)\mu(t,x) \ in \ ]0,T[\times I \\ \gamma y = 0 \ in \ ]0,T[\times \Gamma \\ y(0,x) = y_0(x) \ in \ I \end{cases} \tag{2.6}$$

We set the operator A defined by

$$D(A) = \left\{ u \in H_0^1(I) / -\frac{d^2u}{dx^2} \in L^2(I) \right\}; \forall u \in D(A), \ Au = -\frac{d^2u}{dx^2}$$

According to the spectral theory Brezis [10], Hörmander [11], A admits a Hilbert base of $L^2(I)$ of eigenfunctions $(w_k)_{k\geq 1}$ whose associated eigenvalues are $(\lambda_k)_{k\geq 1}$ rows in the decreasing direction where

$$\begin{cases} w_k(x) &= \sqrt{\frac{2}{\pi}} sin(kx) \quad and \\ \lambda_k &= k^2 \end{cases}$$

See also Lebeau [7] and Fursikov [6].

Definition 2.1. An integrable square function $\mu : I \subset \Omega \longrightarrow \bar{\mathbb{R}}$ is called strategic (see El. Jai[9]) if it satisfies,
for all $\phi_0 \in L^2(I)$, the solution ϕ^+ of heat equation

$$\begin{cases} {\phi^+}'(t,x) - \triangle\phi^+(t,x) = 0 \ in \ Q_T =]0, +\infty[\times I \\ \gamma\phi^+(t,x) = 0 \ in \ \Sigma_T =]0, +\infty[\times \partial I \\ \phi^+(0) = \phi_0 \ in \ I \end{cases} \tag{2.7}$$

satisfies:

$$\forall\ t > 0,\ \int_I \mu(t,x)\phi^+(t,x)dx = 0 \qquad \text{then} \qquad \phi_0 = 0. \tag{2.8}$$

Remark 2.2.

1. In 1 dimension (see Seck [8] and Ane [12]), it has been proved that there are internal strategic actuators, this is the case of certain particular geometries.
2. Similarly, the Exact controllability border of the heat equation in 1D by strategic actuators area by the SCD Method(surjectivity by compactness and density) was done see Seck [8] and Jai [7].

3 Internal Temporally Strategic Actuator

Definition 3.1. An integrable square function

$$\mu : \begin{pmatrix} \mathbb{R}_+^* \times I \to \bar{\mathbb{R}} \\ (t,x) \mapsto \mu(t,x) \end{pmatrix}.$$

is called temporally strategic if for all $\phi_0 \in L^2(I)$ such that

$$\forall t \in]0,+\infty[, \qquad \int_I \mu(t,x)\phi^+(t,x)dx = 0 \tag{3.1}$$

then $\phi_0 = 0$.

Remark 3.1. If the application

$$t \mapsto \int_I \mu(t,x)\phi^+(t,x)dx \tag{3.2}$$

is analytic, it is enough that the relation is true for all $t \in]0,T[$ so that it is true for all $t \in \mathbb{R}_+^*$

An example of a temporally strategic actuator is given:

Theorem 3.1. *If $\phi_0 \in L^2(I)$ and verify: $\forall k \in \mathbb{N}^*$, $\phi_{0k} \neq 0$ for all open non-empty $\mathcal{O}$ content in I then $\mu(t,x) = \chi_{\mathcal{O}}\phi^+(t,x)$ is a temporarily strategic actuator.*

Proof. We have:

$$\begin{aligned} \mu_k(t,x) &= \int_I \chi_{\mathcal{O}}\phi^+(t,x)w_k(x)dx \\ &= \int_{\mathcal{O}} \phi^+(t,x)w_k(x)dx \\ &= \sum_{h=1}^{+\infty} \phi_{0h}e^{-\lambda_h t}\int_{\mathcal{O}} w_h w_k dx \end{aligned}$$

Assume $\alpha_{kh} = \int_{\mathcal{O}} w_h w_k dx$ we obtain

$$\mu_k(t,x) = \sum_{h=1}^{+\infty} \phi_{0h}\alpha_{kh}e^{-\lambda_h t}$$

Let $\zeta_0 \in L^2(I)$ such that:

$$\forall t > 0, \qquad \int_I \mu(t,x)\zeta^+(t,x)dx = 0 \text{ then } \sum_{h=1}^{+\infty} \zeta_{0k}\mu_k(t,x)e^{-\lambda_k t} = 0 \tag{3.3}$$

let

$$\forall t > 0, \qquad \sum_{k=1}^{+\infty}\sum_{h=1}^{+\infty} \zeta_{0k}\phi_{0h}(t)\alpha_{kh}e^{-(\lambda_k+\lambda_h)t} = 0 \tag{3.4}$$

so $\zeta_{0k}\phi_{0h}(t)\alpha_{kh} = 0\ \forall h, k$.
As $\phi_{0h} \neq 0$ then $\forall h, \forall k\ \zeta_{0k}\alpha_{kh} = 0$.
Suppose it exists $k_0 \in \mathbb{N}^*$ such that $\forall h \in \mathbb{N}^*\ \alpha_{hk_0} = 0$.
Also

$$\forall h \in \mathbb{N}^*, \qquad \int_I \chi_{\mathcal{O}} w_{k_0}(x) w_h(x) dx = 0 \tag{3.5}$$

so $\chi_{\mathcal{O}} w_{k_0}$ is identically null.
Or w_k is a non-zero eigen function, this is impossible ■ □

Corollary 3.2. *For any open non-empty contained in I, there is a temporarily strategic μ actuator whose support contained in $\mathbb{R}_+^* \times \mathcal{O}$.*

Proof. Just take $\omega \subseteq \mathcal{O}$, $\phi_0 \in L^2(I)$ such that $\forall k \in \mathbb{N},\ \phi_{0k} \neq 0$ and, to pose $\mu(t,x) = \chi_{\mathcal{O}}\omega(x)\phi^+(t,x)$
■ □

4 Main Result: Exact Internal Controllability of the Heat Equation

4.1 Some functional analysis notes

It is recalled that the exact controllability is reflected here by this definition:

Definition 4.1. (reminder) The L operator defined under (4.3). The system defined by (4.6) will be say **exactly controllable on time T** if and only if L is surjective (see Ramdani [13, 14]).

We define the operator ϕ_τ, which to a command $v \in L^2(0,\tau;\mathcal{O}) \subset L^1(0,\tau;\mathcal{O})$ associates the solution z solution of the system (4.6) at the moment τ:

$$\begin{aligned} \phi_\tau : L^2(0,\tau;\mathcal{O}) &\longrightarrow X \\ v &\mapsto z(\tau) = \int_0^\tau e^{(\tau-s)A}\beta v(s)ds \end{aligned} \tag{4.1}$$

it's clear that $\phi_\tau \in \mathcal{L}(L^2(0,\tau;\mathcal{O}), X)$, since there is a constant $k_\tau > 0$ dependent only of τ, of A and of β such that

$$||\phi_\tau(v)|| \leq k_\tau ||v||_{L^2(0,\tau;\mathcal{O})}, \qquad \forall v \in L^2(0,\tau;\mathcal{O}) \tag{4.2}$$

Moreover, the adjunct linear operator ϕ_τ^* of ϕ_τ is given by the following proposition

Proposition 4.1. *Let $\phi_\tau \in \mathcal{L}(L^2(0,\tau;X))$ the operator defined by system (4.6). His adjunct $\phi_\tau^* \in \mathcal{L}(X, L^2(0,\tau;X))$ is the operator*

$$\begin{aligned} \phi_\tau^* : X &\longrightarrow L^2(0,\tau;\mathcal{O}) \\ z_0 &\mapsto v \end{aligned} \tag{4.3}$$

where $v(t) = \beta^* e^{-(\tau-t)A} z_0$, $\forall t \in (0,\tau)$

Proof. Since ϕ_τ is a bounded operator, it suffices to notice that for all $v \in L^2(0, \tau; X)$ and for all $z_0 \in X$, we have (since $A^* = -A$)

$$(\phi_\tau, z_0) = \int_0^\tau \left(e^{(\tau-s)A}\beta v(s), z_0\right) ds = \int_0^\tau \left(v(s), \beta^* e^{-(\tau-s)A} z_0\right)_{\mathcal{O}} ds = (v, w)_{\left(L^2(0,\tau);\mathcal{O}\right)}$$

so that we have $w = \phi_\tau^* z_0$. □

Let the following two fundamental lemmas

Lemma 4.1. *Let Y, Z be two Hilbert spaces and $G \in \mathcal{L}(Y, Z)$. So, G is surjective if and only its adjunct G^* is lower bounded, ie there exist $C > 0$such that*

$$||G^* z||_Y \geq C||z||_Z, \qquad \forall z \in Z \tag{4.4}$$

Proof. It suffices to apply proposition4.1 below by taking $X = Z$ and as operator F the identity over Z. □

Lemma 4.2. *Given three Hilbert spaces X, Y, Z, let $F \in \mathcal{L}(X, Z)$ and $G \in \mathcal{L}(Y, Z)$. So, the following assertions are equivalent:*

i. $ImF \subset ImG$.

ii. There exist a constant $C > 0$ such that

$$||G^* z|| \geq C||F^* z||_X, \qquad \forall z \in Z. \tag{4.5}$$

iii. There exist $H \in \mathcal{L}(X, Y)$ such that $F = GH$.

Proof. See Karim Ramdani [13]. □

4.2 Main result

Theorem 4.3. *If μ is a temporally strategic actuator, for all $y_0 \in F_T^{0,-1}$, there exists $\beta \in L^2(]0, T[)$ such that the solution y of the equation*

$$\begin{cases} y'(t, x) - \triangle y(t, x) = \beta(t)\mu(t, x) \ in \ ]0, T[\times I \\ \gamma y = 0 \ in \ ]0, T[\times \Gamma \\ y(0, x) = y_0(x) \ in \ I \end{cases} \tag{4.6}$$

verify $y(T) = 0$.

Proof. **Fisrt step:**

Formal construction of the L operator?

We first give a notation: If $f \in L^2(]0, +\infty[\times I)$ and for t fixed, we write $f^+(s, t, x)$ the solution of the direct homogeneous heat equation of the initial data $f(t, x)$. So we have:

$$f^+(s, t, x) = \sum_{k=1}^{+\infty} f_k(t) e^{-\lambda_k s} w_k(x) \tag{4.7}$$

where $(f_k)_{x\geq 1}$ is given by $f(t, x) = \sum_{k=1}^{+\infty} f_k(t) w_k$.

In the following, we will also use the functional notation $f^+(s,t)$.
Let $\psi_0 \in F_T^{0,1}$, multiply the equation (4.6) by ψ^+ and integrating by parts:

$$\int_I y(T)\psi(T)dx - < y_0, \psi_0 >_{F_T^{0,-1}, F_T^{0,1}} = \int_0^T \int_I \mu(t,x)\psi(t,x)dxdt$$

let

$$\begin{aligned} < y_0, \psi_0 >_{F_T^{0,-1}, F_T^{0,1}} &= -\int_0^T \beta(t) < \mu(t,x), \psi_0 >_{F_T^{0,-1}, F_T^{0,1}} dt \\ &= - < \int_0^T \beta(t)\mu(t,x)dt, \psi_0 >_{F_T^{0,-1}, F_T^{0,1}} \end{aligned}$$

from where

$$y_0 = -\int_0^T \beta(t)\mu(t,x)dt$$

so

$$L_1(\beta) = -\int_0^T \beta(t)\mu(t,x)dt$$

We are posing now: $L(\beta) = -\left(L_1(\beta)\right)^+(T)$ so

$$L(\beta) = -\int_0^T \beta(t)\mu^+(T-t,x)dt \tag{4.8}$$

Now apply the following theory to the F_T and F_T^* spaces by taking:

$$F_T^* \overset{Id}{\longmapsto} F_T^*$$

and

$$L^2(]0;T[) \overset{L_1}{\longmapsto} F_T^*$$

then

$$F_T^* \subset Im(L) \Longleftrightarrow \forall z^* \in F_T, \text{ there exist } C > 0 \text{ such that } |z| \leq C.||L^*(z)||$$

Thus the operator L^* is bounded below.

Second step.

$L \in \mathcal{L}\left(L^2(]0,T[), L^2(I)\right)$?
Let $\beta \in L^2(]0,T[)$, we have

$$\begin{aligned} \sum_{k=1}^{+\infty}\left(\int_0^T \beta(t)\mu_k(t,x)e^{-\lambda_k(T-t)}dt\right)^2 &\leq \sum_{k=1}^{+\infty} ||\beta||_{L^2(]0,T[)} \int_0^T \mu_k^2(t)e^{-2\lambda_k(T-t)}dt \\ &\leq \sum_{k=1}^{+\infty} ||\beta||_{L^2(]0,T[)} ||\mu_k||^2_{L^2(]0,T[)} \\ &\leq ||\beta||_{L^2(]0,T[)} ||\mu||^2_{L^2(]0,T[\times I)} \end{aligned}$$

$L(\beta) \in L^2(I)$ and

$$||L(\beta)|| \leq ||\beta||_{L^2(]0,T[)} ||\mu||^2_{L^2(]0,T[\times I)}. \tag{4.9}$$

Third step:

$L(L^2(]0,T[))$ is dense in $L^2(I)$?

Let $\psi_0 \in F_T$ such that $\psi(T)$ is orthogonal to $L(L^2(]0,T[))$; let $\beta \in L^2(]0,T[)$, we have

$$(L(\beta), \psi(T))_{L^2(I)} = 0$$

so

$$\begin{aligned}
\int_I \int_0^T \beta(t)\mu^+(T-t,x)\psi(T)dxdt &= 0 \Longrightarrow \\
\int_0^T \beta(t) \int_I \mu^+(T-t,x)\psi(T)dxdt &= 0 \Longrightarrow \\
\int_0^T \beta(t) \int_I \mu(t,x)\psi(t)dxdt &= 0
\end{aligned}$$

from where

$$\forall t \in]0,T[, \ \int_I \mu(t,x)\psi(t)dtdx = 0$$

so $\psi_0 \equiv 0$ from where $\psi_0(T) = 0$.

Fourth step:

L^*L is compact of $L^2(]0,T[)$ in $L^2(]0,T])$?

$$\begin{aligned}
(L(\beta), L(\gamma))_{L^2(I)} &= \int_\Omega \int_0^T \beta(t)\mu^+(T-t,x)dt \int_0^T \gamma(s)\mu^+(T-s,x)dsdx \\
&= \int_0^T \int_0^T \beta(s)\gamma(t) \int_I \mu^+(T-t,x)dxdsdt \\
&= \int_0^T \gamma(t) \int_0^T \beta(s) \left(\int_I \mu^+(T-s,x)dx \right) dsdt
\end{aligned}$$

so

$$\begin{aligned}
L^*L(\beta)(t) &= \int_0^T \beta(s) \int_I \mu^+(T-s,x)\mu^+(T-t,x)dxds \\
&= \int_0^T \beta(s) \sum_{k=1}^{+\infty} \mu_k(t,x)\mu_k(s,x)e^{-\lambda_k(T-s)}e^{-\lambda_k(T-t)}ds \\
&= \int_0^T \beta(s)K(t,s)ds
\end{aligned}$$

where

$$K(t,s) = \sum_{k=1}^{+\infty} \mu_k(t,x)\mu_k(s,x)e^{-\lambda_k(T-s)}e^{-\lambda_k(T-t)}$$

K is continuous on $[0,T] \times [0,T]$, just take $\mu(t,x)$ regular and the choice of ϕ_0 regulates the problem.

Step five: Conclusion

Let $y_0 \in F_T^{0,-1}$ then it exists $\beta \in L^2(]0,T[)$ such that

$$y^+(T) = -\int_0^T \beta(t)\mu^+(T-t,x)dt$$

Let $\psi_0 \in F_T^{0,1}$, we have

$$\begin{aligned} \langle y^+(T), \psi(T) \rangle_{L^2(I)} &= \langle y_0, \psi_0 \rangle_{F_T^{0,-1}, F_T^{0,1}} \\ &= -\int_I \left(\int_0^T \mu^+(T-t,x)dt \right) \psi(T)dx \\ &= -\int_0^T \int_I \mu(t,x)\psi(t)dxdt \end{aligned}$$

On the other hand, there is the solution of

$$\begin{cases} y' - \triangle y = \beta(t)\mu(t,x) \; in \;]0,T[\times I \\ \gamma y = 0 \; in \;]0,T[\times \Gamma \\ y(0) = y_0 \; in \; I \end{cases} \tag{4.10}$$

then

$$\int_I y(T)\psi(T)dx - \langle y_0, \psi_0 \rangle_{F_T, {}^{0,-1}, F_T^{0,1}} = \int_0^T \int_I \mu(t,x)\beta^-(t)dxdt$$

so

$$\int_I y(T)\psi(T)dx = 0$$

from where $y(T) = 0$. ■ □

Remark 4.1. We can choose the control as regular as we want.

Theorem 4.4. *Let $\mathcal{O}$ an open part of I not empty, for all $y_0 \in F_T^{0,-1}$, it exists $\mu \in L^2(]0,T[\times I)$ to support contained in $]0,T[\times \mathcal{O}$ such that if y is solution*

$$\begin{cases} y' - \triangle y = \beta(t)\mu(t,x) \; in \;]0,T[\times I \\ \gamma y = 0 \; in \;]0,T[\times \Gamma \\ y(0) = y_0 \; in \; I \end{cases} \tag{4.11}$$

then $y(T) = 0$.

Proof. Take η functions to support contained in $\mathcal{O}$, belonging to $D(I)$ non-zero on an open content in $\mathcal{O}$.

Let $\psi_0 \in D(A)$ such that $\forall k \in \mathbb{N}^*, \phi_{0k} \neq 0$ then $\mu(t,x) = \eta\phi^+(t)$ is temporally strategic and we apply the theorem 4.3 ■ □

5 Conclusion and Perspective

In this work, we established the exact controllability of the heat equation in any finite dimension by the use of strategic zone actuators.

a Thus, by a new approach called (Surjectivity by compactness and Density), the internal exact controllabilty of the 1D heat equation by the use of strategic zone actuators was done.

b Indeed, by the construction of a compact surjective linear operator resulting from the system we have succeeded in achieving the exact internal controllability of the heat equation in any dimension by using the notion of strategic actuators.

c In the near future, we would want to extend these results in non-convex domains, with cracks and with corners. Also, we would to investigate others kind linear EDP. For example explore the case of equations in evolutions of wave type, Schrödinger, plates and Navier Stokes linearized.

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Competing Interests

Authors have declared that no competing interests exist.

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